Restricted Turing Machines and Language Recognition

Giovanni Pighizzini

Dipartimento di Informatica
Università degli Studi di Milano, Italy

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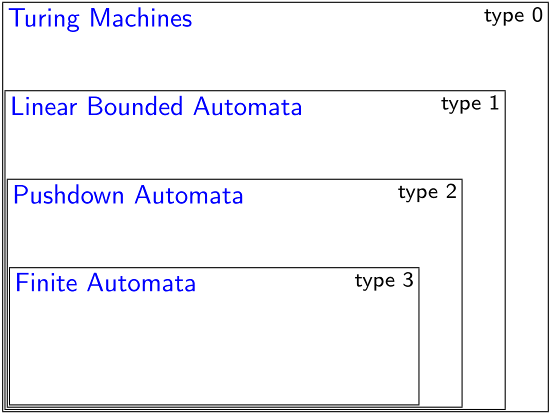
Introduction

The Chomsky Hierarchy



Limited Automata [Hibbard '67]

One-tape Turing machines with restricted rewritings

Definition

Fixed an integer $d \geq 1$, a *d-limited automaton* is

- a one-tape Turing machine
- which is allowed to rewrite the content of each tape cell *only in the first d visits*

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Computational power

- For each $d \geq 2$, d-limited automata characterize context-free languages [Hibbard '67]

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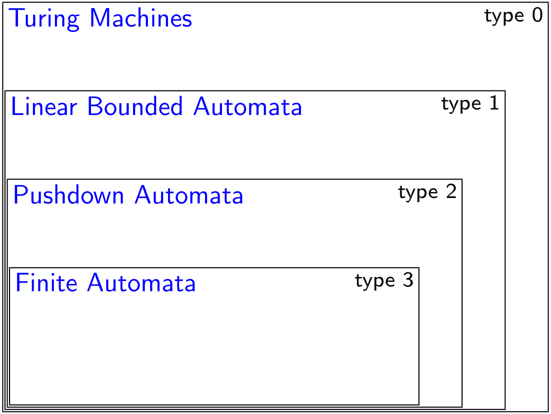

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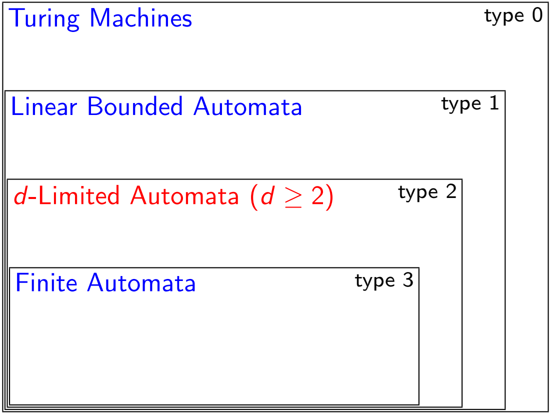

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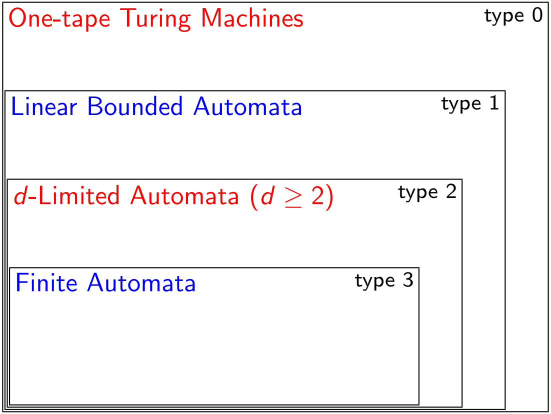

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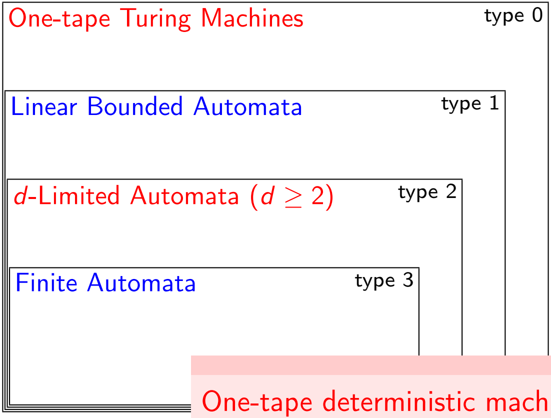

One-tape deterministic machines working in *linear time* can recognize only regular languages [Hennie '65]

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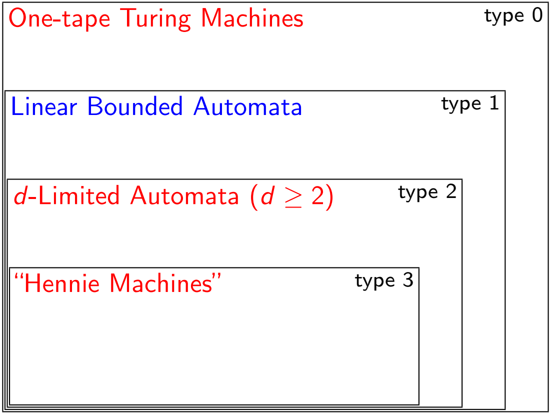

General Contents

Part I: Fast One-Tape Turing Machines

Hennie Machines & C

Part II: One-Tape Turing Machines with Rewriting Restrictions

Limited Automata & C

Outline

- One-Tape Turing machines
- Time complexity: different measures
- Crossing sequences
- Lower bounds for nonregular languages recognition
- Optimality
- Fast recognition of unary nonregular languages
- Final remarks: other complexity measures

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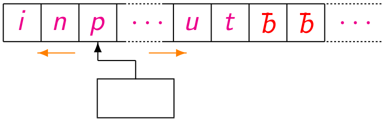

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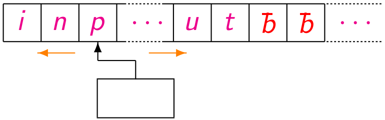

- *Finite state control*
- *Semi-infinite tape*

  at the beginning:

  - input string (leftmost part)
  - blank symbol (remaining squares)

- *Computation step*
  - change of state
  - nonblank symbol written in the scanned tape cell
  - head moved either to the left, or to the right, or kept on the same cell

- *Accepting* and *rejecting states*: the computation stops

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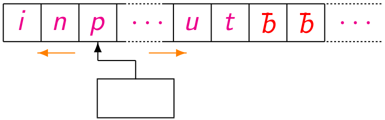

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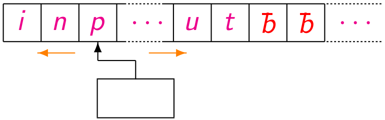

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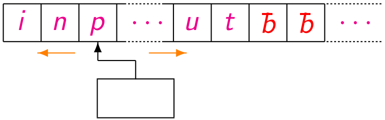

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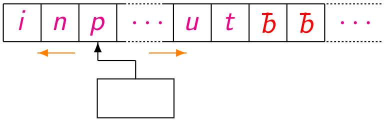

- Deterministic version (dTM)

- Nondeterministic version (nTM)

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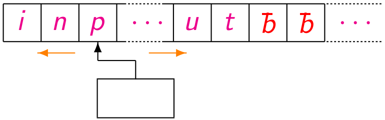

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- $t(\mathcal{C})$ number of moves in the computation $\mathcal{C}$

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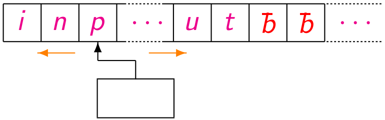

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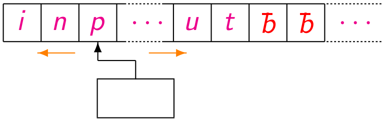

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Nondeterministic case: several computations on a same input

How to define $t(x)$ and $t(n)$?

Complexity Measures

strong measure: costs of *all computations* on x

$$t(x) = \max\{t(\mathcal{C}) \mid \mathcal{C} \text{ is a computation on } x\}$$

worst case!

weak measure: *minimum* cost of *accepting* x

$$t(x) = \begin{cases} \min\{t(\mathcal{C}) \mid \mathcal{C} \text{ is accepting on } x\} & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}$$

best case for acceptance!

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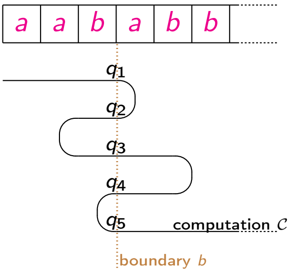
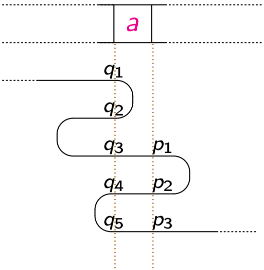
Crossing Sequences



Crossing sequence of a computation $\mathcal{C}$
at a boundary b between two tape squares:

- $(q_1, \ldots, q_k)$
- q_i is the state when b is crossed for the ith time

Crossing Sequences: Compatibility



- $(q_1, \ldots, q_k)$, $(p_1, \ldots, p_h)$: finite crossing sequence
- it is possible to verify whether or not they are *compatible* with respect to an input symbol a,

  i.e., $(q_1, \ldots, q_k)$ and $(p_1, \ldots, p_h)$ could be at the left boundary and at the right boundary of a tape square which initially contains the symbol a

Lower Bounds

One-Tape Machines

Problem:

Find tight lower bounds for

- the minimum amount of time $t(n)$
- the length of crossing sequences $c(n)$

for nonregular language recognition

Theorem

If L is accepted by a nTM such that $c(n) = O(1)$, under the weak *measure, then L is regular*

Proof idea:

- Let K be such that $c(n) \leq K$
- Define a NFA A accepting L s.t.
  - the states are the crossing sequences of length $\leq K$
  - the transition function is defined according to the "compatibility" between crossing sequences

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One-Tape Machines: Simple Bounds

Time

Theorem

If L is accepted by a nTM such that $t(n) = o(n)$, under the weak *measure, then $t(n) = O(1)$ and L is regular*

Proof idea:

- Let n_0 s.t. $t(n) < n$, for each $n \geq n_0$
- Given $x \in L$ with $|x| \geq n_0$, there is a computation $\mathcal{C}$ that accepts x just reading a proper prefix x' of length $\leq t(x)$
- $\mathcal{C}$ should also accept x'
- Since all x' is read in $\mathcal{C}$, $t(x') \geq x'$ implying $|x'| < n_0$
- Hence, the membership to L can be decided just testing an input prefix of length at most n_0

Remark: The same argument works for multitape machines

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Does it is possible to improve the lower bounds
on $c(n)$ and $t(n)$ for nonregular language recognition
given in the previous results?

Different bounds have found depending

- on the measure (strong, accept, weak)
- on the kind of machines (deterministic, nondeterministic)

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Deterministic Machines: Lower Bounds (strong measure)

- Hennie (1965) proved that

  one-tape *deterministic* machines working in linear time
  accept only regular languages

  Furthermore, in order to accept nonregular languages

  $c(n)$ must grow at least as $\log n$

- Trakhtenbrot (1964) and Hartmanis (1968), independently, got a better time lower bound:

  in order to recognize a nonregular language a dTM needs
  time $t(n)$ growing at least as $n \log n$

- Optimal!
  There are nonregular languages accepted in time $O(n \log n)$

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Nondeterministic Machines

weak measure:

- There is nonregular language accepted by a nTM in $o(n \log n)$ time [Wagner&Wechsung '86]
- There is a NP-complete language accepted by a nTM in $O(n)$ time [Michel '91]

strong measure:

- The time lower bound $n \log n$ proved for dTMs also holds for nTMs [Tadaki&Yamakami&Lin '10]

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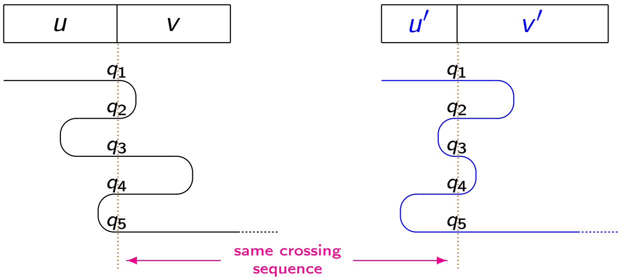

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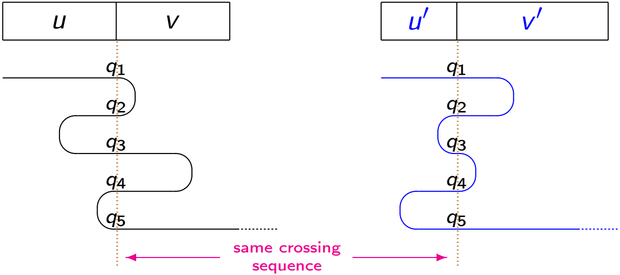

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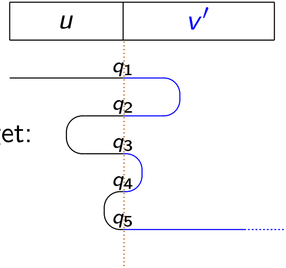

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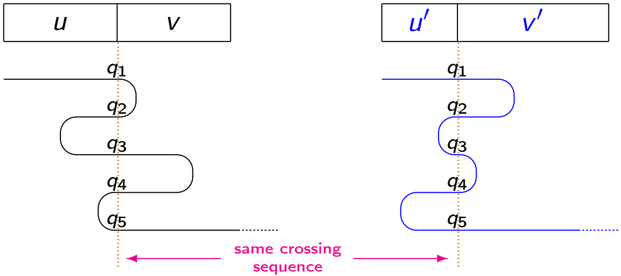

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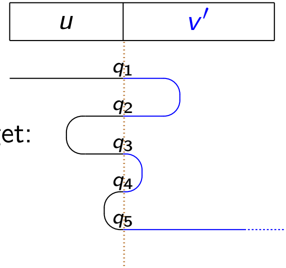

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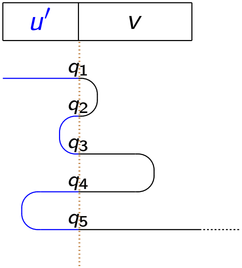

Lower Bounds for Accept Measure

Lemma

If a string w is accepted by a computation $\mathcal{C}$ having a same crossing sequence at 3 different boundaries of the input, then there is a computation $\mathcal{C}'$ with $c(\mathcal{C}') = c(\mathcal{C})$ accepting a shorter string w'

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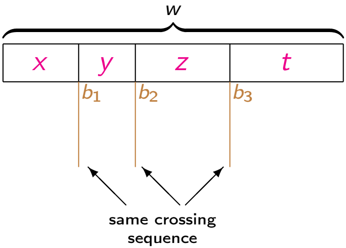

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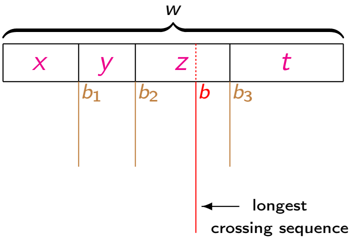

case $b \geq b_2$

Lower Bounds for Accept Measure

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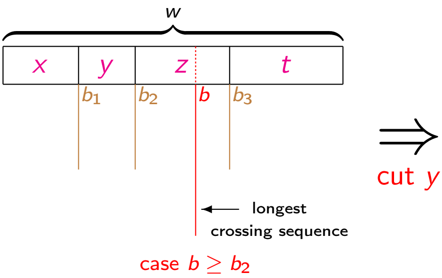

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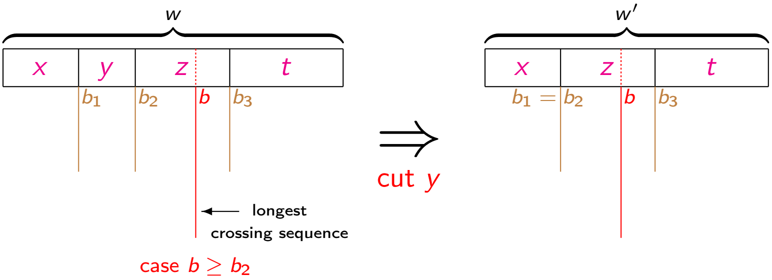

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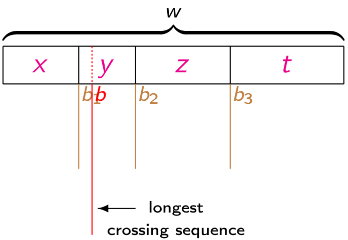

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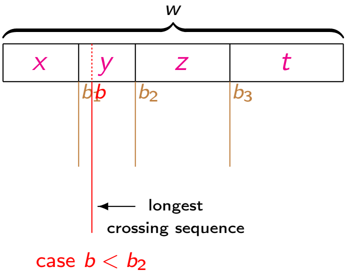

$\Longrightarrow$ cut z

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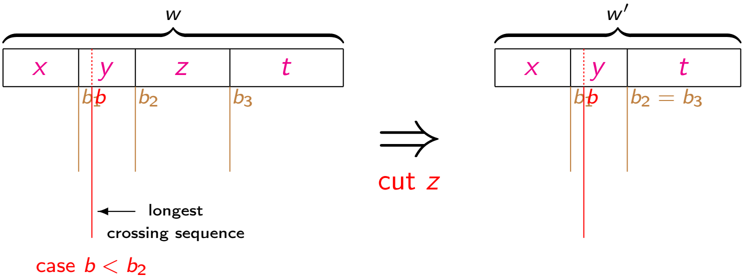

Accept Measure: Lower Bound for $c(n)$

- $L[k] :=$ set of strings having an accepting computation $\mathcal{C}$ with $c(\mathcal{C}) = k$
- $w_k :=$ a shortest string in $L[k]$, for $L[k] \neq \emptyset$
- $n_k := |w_k|$
- On w_k each crossing sequence can appear at most twice
- At least $\lfloor \frac{n_k - 1}{2} \rfloor$ different crossing sequences
- Hence $q^{k+1} \geq \lfloor \frac{n_k - 1}{2} \rfloor$ ($q :=$ number of states), then:

$$c(n_k) \geq k \geq \log_q \lfloor \frac{n_k - 1}{2} \rfloor - 1$$

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$c(n) = o(\log n)$ implies $c(n) = O(1)$ and L regular

Accept Measure: Lower Bound for $t(n)$

If $c(n) \neq O(1)$:

- w_k is accepted by a computation $\mathcal{C}_k$ using at least $\lfloor \frac{n_k - 1}{2} \rfloor$ different crossing sequences
- $\mathcal{C}_k$ has $\geq \lfloor \frac{n_k - 1}{4} \rfloor$ crossing sequences of length $\geq \log_q \lfloor \frac{n_k - 1}{4} \rfloor$ (combinatorial argument)
- Hence $\mathcal{C}_k$ consists of at least

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Hence:

$t(n) = o(n \log n)$ implies $c(n) = O(1)$ and, thus,
L regular and $t(n) = O(n)$

Lower Bounds for Accept Measure

Summing up:

Theorem ([P.'09])

Let M be a nTM accepting a language L such that in each accepting computation

- *the length of crossing sequences is bounded by $c(n)$*
- *the time is bounded by $t(n)$*

If $c(n) = o(\log n)$ then $c(n) = O(1)$ and L is regular

If $t(n) = o(n \log n)$ then

- *$t(n) = O(n)$*
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Does it is possible to extend the lower bounds
from the accept to the weak measure?

Time: negative answer

Theorem ([Michel '91])

There exists an NP-complete language accepted in time $O(n)$ by a nTM under the weak *measure*

- Linear time is necessary for regular languages

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- $L :=$ language accepted by the given machine M
- For each $n \geq 1$:

  $N_n :=$ NFA with states all crossing sequences of length $\leq c(n)$,
  transitions defined according to the "compatibility" relation,
  at most $q^{c(n)+1}$ states
- N_n agrees with M on strings of length $\leq n$
- $A_n :=$ DFA equivalent to N_n, at most $2^{q^{c(n)+1}}$ states
- If L is not regular, then
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- Hence $2^{q^{c(n)+1}} \geq \frac{n+3}{2}$, implying

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Summary of the Lower Bounds

		strong	accept	weak
dTM	$t(n)$			
	$c(n)$			
nTM	$t(n)$			
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		strong	accept	weak
dTM	$t(n)$	$n \log n$		
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Trakhtenbrot (1964) and Hartmanis (1968)
Hennie (1965) for $c(n)$

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Tadaki, Yamakami, and Lin (2010)

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Pighizzini (2009)

Summary of the Lower Bounds

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Consequence of accept nondeterministic case

Summary of the Lower Bounds

		strong	accept	weak
dTM	$t(n)$	$n \log n$	$n \log n$	$n \log n$
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For deterministic machines, accept and weak is the same

Summary of the Lower Bounds

		strong	accept	weak
dTM	$t(n)$	$n \log n$	$n \log n$	$n \log n$
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nTM	$t(n)$	$n \log n$	$n \log n$	n
	$c(n)$	$\log n$	$\log n$	$\log \log n$

$t(n)$: simple bound
$c(n)$: Pighizzini (2009)

Summary of the Lower Bounds

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Optimality

Optimality of the Bounds

$L = \{a^{2^m} \mid m \geq 0\}$ [Hartmanis '68]

L is accepted by a dTM M as follows:

- At the beginning all the input cells are "unmarked"
- M sweeps form left to right over the input segment and marks off the 1st, 3th, 5th, etc. unmarked squares
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- At the beginning all the input cells are "unmarked"
- M sweeps form left to right over the input segment and marks off the 1st, 3th, 5th, etc. unmarked squares
- M repeats the previous step until the rightmost square of the input segment becomes marked
- M accepts if and only if all the input segment is marked

a	a	a	a	a	a	a	a	a	a	a	a	input a^{12}

Optimality of the Bounds

$L = \{a^{2^m} \mid m \geq 0\}$ [Hartmanis '68]

L is accepted by a dTM M as follows:

- At the beginning all the input cells are "unmarked"
- M sweeps form left to right over the input segment and marks off the 1st, 3th, 5th, etc. unmarked squares
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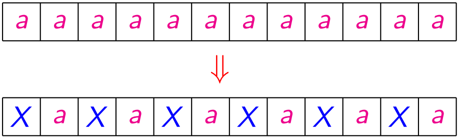

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X	a	X	a	X	a	X	a	X	a	X	a

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X	a	X	a	X	a	X	a	X	a	X	a

$$\Downarrow$$

X	X	X	a	X	X	X	a	X	X	X	a

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X	X	X	a	X	X	X	a	X	X	X	a

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X	X	X	X	X	X	X	a	X	X	X	X

reject!

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a	a	a	a	a	a	a	a

input a^8

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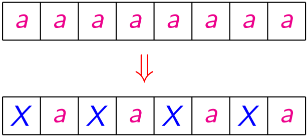

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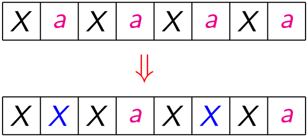

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X	X	X	X	X	X	X	X

accept!

Complexity

- On input a^n, M makes $O(\log n)$ sweeps of the tape:

$$c(n) = O(\log n) \text{ and } t(n) = O(n \log n)$$

- M is *deterministic* and the previous bounds are satisfied by *all* computations: strong measure

- This gives the optimality of all the lower bounds in the table

		strong	accept	weak
dTM	$t(n)$	$n \log n$	$n \log n$	$n \log n$
	$c(n)$	$\log n$	$\log n$	$\log n$
nTM	$t(n)$	$n \log n$	$n \log n$	
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with the exception of those for nTMs, under the weak measure

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Weak Measure: Optimality for nTMs

- There are nonregular languages accepted in time $O(n)$ [Michel '91]
- Regular languages require time n
- No "gap" between regular and nonregular languages
- The example in [Michel '91] strongly relies on an input alphabet with *more than one symbol*
- Up to now, we do not know any example of unary nonregular language accepted in weak time $O(n)$
- *Conjecture:* If a nTM accepts a *unary* language L in time $o(n \log \log n)$ under the weak measure then L is regular
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Fast Recognition of Unary Languages

nTMs and Unary Languages: Basic Techniques

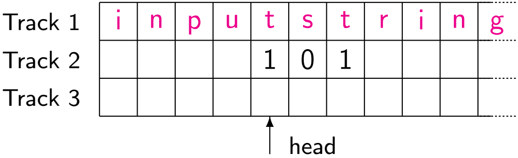
Tape Tracks

- We can consider a tape divided in a fixed number of tracks
- The input is written on the first track

Track 1	i	n	p	u	t	s	t	r	i	n	g
Track 2	m	e	m	o	r	y	s	p	a	c	e
Track 3	m	e	m	o	r	y	s	p	a	c	e

nTMs and Unary Languages: Basic Techniques

How to count input symbols



- A counter is kept on track 2,
  starting from the position scanned by the tape head
- When the head is moved to the right,
  the counter is incremented to count one more position
  and it is shifted to the right
- This is done in $O(\log j)$ steps using track 3 as an auxiliary variable (j = value of the counter)
- k tape positions are counted in $O(k \log k)$ moves

nTMs and Unary Languages: Basic Techniques

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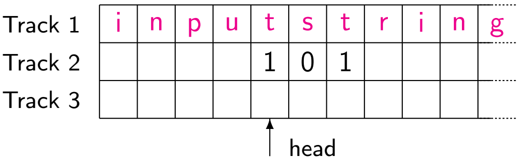

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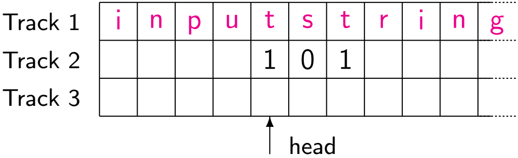

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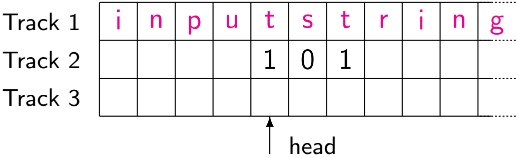

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nTMs and Unary Languages: Basic techniques

How to compute n mod k

$n =$ input length
$k =$ integer written somewhere

- Reset the counter on track 2 each time it becomes equal to k
- When the end of the input is reached, track 2 contains $n \bmod k$
- To implement the comparison between the counter and k:
  - The value of k is kept on an extra track (track 4)
  - When the input head is moved to the right to count one more position, the representation of k is moved one position to the right in such a way that it is always aligned with the counter on track 2, to make easy the comparison
- The total time is $O(n \log k)$

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A Unary Language Accepted in Weak Time $O(n \log \log n)$

- For each integer n let

$$q(n) := \text{ the smallest integer that does not divide } n$$

- We consider the language

$$L = \{a^n \mid q(n) \text{ is not a power of } 2\}$$

- L is recognized by the following nondeterministic algorithm:

[Mereghetti'08]

input a^n
guess an integer s, $s > 1$
guess an integer t, $2^s < t < 2^{s+1}$
if $n \bmod 2^s = 0$ **and** $n \bmod t \neq 0$ **then** accept
else reject

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Implementation and complexity:

- Two extra tracks (track 5 and 6) are used to guess 2^s and t (linear time)

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- Two extra tracks (track 5 and 6) are used to guess 2^s and t (linear time)
- Using the previous technique, $n \bmod 2^s$ and $n \bmod t$ are computed (time $O(n \log t)$)

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A Unary Language Accepted in Weak Time $O(n \log \log n)$

$$L = \{a^n \mid \text{ the smallest integer which does not divide } n \text{ is not a power of } 2\}$$

We have proved the following:

Theorem ([P.'09])

L is accepted by a one-tape nondeterministic machine with

- $t(n) = O(n \log \log n)$
- $c(n) = O(\log \log n)$

under the weak *measure*

More on L: Space Complexity

The language L and its complement have been widely studied in the literature.

Some results concerning space:

- L^c is accepted by a dTM with a separate worktape, using the minimum amount of space $O(\log \log n)$
  [Alt&Mehlhorn '75]

- For L we can even do better:
  L is accepted by a *one-way* nTM with a separate worktape, using the minimum amount of space $O(\log \log n)$, under the weak measure [Mereghetti '08]

L seems a good example of nonregular language with "low" complexity

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Final Remarks

Closing Remarks

We considered the "border" between regular and nonregular languages, wrt to the time $t(n)$ and the length of crossing sequences $c(n)$

Similar investigations can be have been done (even for different classes of languages) wrt other resources:

- Space e.g. [Sziepietowski '94, Mereghetti '08]
- Head reversals e.g. [Bertoni&Mereghetti&P.'94]
- Return complexity or Active visits [Wechsung '75, Chytil '76]
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