Investigations on Automata and Languages over a Unary Alphabet

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Unary or Tally Languages

- One letter alphabet $\Sigma = \{a\}$
- Many differences with the general case have been discovered

  First example:

  Theorem [Ginsurg&Rice '62]

  Each unary context-free languages is regular

- Structural complexity: classes of *tally sets*
  - Hartmanis, 1972
  - Book, 1974, 1979
  - ...

  Space complexity:
  - Alt&Mehlhorn, 1975
  - Geffert, 1993
  - ...

Unary or Tally Languages

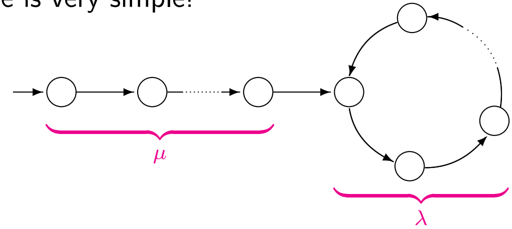
This talk:

- Focus mainly on *descriptional complexity aspects*
  - Optimal simulations between variants of unary automata
  - Unary two-way automata:
    connection with the question $L \stackrel{?}{=} NL$
  - Unary context-free grammars and pushdown automata
- Devices accepting nonregular languages

Unary Automata

Unary One-Way Deterministic Automata (1DFAs)

The structure is very simple!



Theorem

$L \subseteq \{a\}^*$ is regular iff $\exists \mu \geq 0, \lambda \geq 1$ s.t.

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L$$

When $\mu = 0$ the language L is said to be *cyclic*

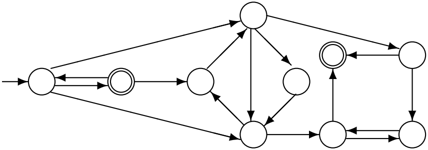
Unary One-Way Nondeterministic Automata (1NFAs)

The structure can be very complicate!

Each direct graph with

- *a vertex selected as initial state*
- *some vertices selected as final states*

is the transition diagram of a unary 1NFA!

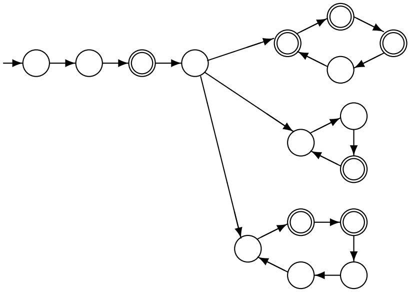


However, we can always obtain an equivalent 1NFA with a

- *simple* and
- *not too big*

transition graph

Chrobak Normal Form for 1NFAs



- An initial *deterministic path*
- Some disjoint *deterministic loops*
- *Only one nondeterministic decision*

Theorem ([Chrobak '86])

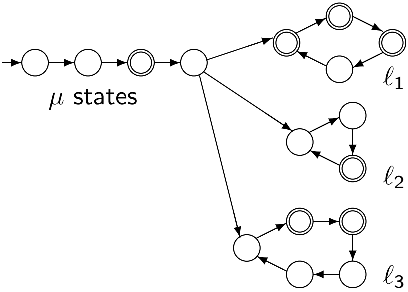
Each unary n-state 1NFA can be converted into an equivalent 1NFA in Chrobak normal form with

- *an initial path of $O(n^2)$ states*
- *total number of states in the loops $\leq n$*

Conversion to Chrobak Normal Form for 1NFAs

- Subtle error in the original proof fixed by To (2009)
- Different transformation proposed by Geffert (2007)
- Polynomial time conversion algorithms
  by Martinez (2004), Gawrychowski (2011), Sawa (2013)
- From the results by Geffert and Gawrychowski:
  - length of the initial path $\leq n^2 - n$
  - total number of states in the loops $\leq n - 1$
    (except when the given 1NFA is the trivial loop of n states)

Removing Nondeterminism from Unary Automata



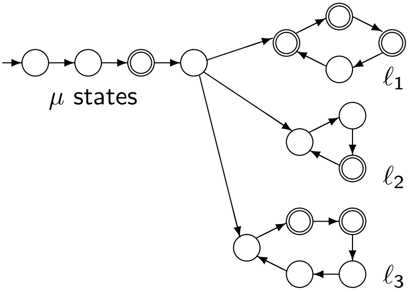
- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a n-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

How large can be $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$?

$$F(n) = \max\{\mathrm{lcm}\{\ell_1, \ell_2, \ldots, \ell_s\} \mid s \geq 1 \wedge \ell_1 + \ell_2 + \cdots + \ell_s \leq n\}$$

Landau's function (1903) $F(n) = e^{\Theta(\sqrt{n \ln n})}$ [Szalay '80]

Removing Nondeterminism from Unary Automata



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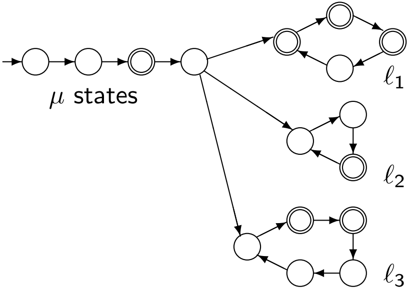
- From a n-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

- $F(n)$ states are also necessary in the worst case [Chrobak '86]

Theorem ([Ljubič '64, Chrobak '86])

The state cost of the simulation of unary n-state 1NFAs by equivalent 1DFAs is $e^{\Theta(\sqrt{n \ln n})}$

From Chrobak Normal Form to Two-Way Automata



- Check if the input is "short" and accepted on the initial path — $\mu + 1$ states
- Check if the input is accepted on the first loop — ℓ_1 states
- Check if the input is accepted on the second loop — ℓ_2 states
- Check if the input is accepted on the third loop — ℓ_3 states

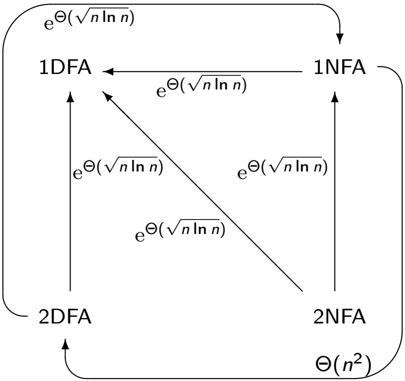
$\mu + \ell_1 + \ell_2 + \cdots + 2$ states are sufficient!

This number is also necessary in the worst case [Chrobak '86]

Theorem

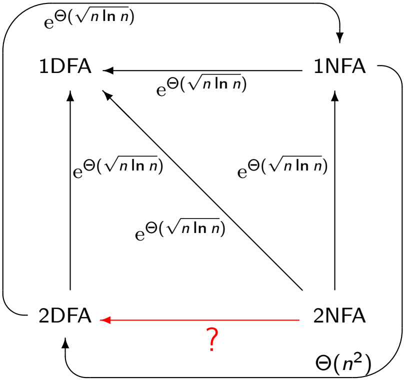
The state cost of the simulation of unary n-state 1NFAs by 2DFAs is $\Theta(n^2)$

Optimal Simulations Between Unary Automata



[Chrobak '86, Mereghetti&P.'01]

Optimal Simulations Between Unary Automata



2NFA → 2DFA Open!

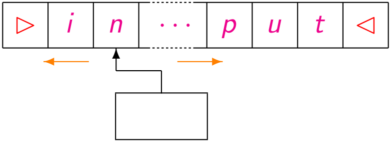
- upper bound $e^{\Theta(\sqrt{n \ln n})}$
  (from 2NFA → 1DFA)
- lower bound $\Omega(n^2)$
  (from 1NFA → 2DFA)

Better upper bound $e^{O(\ln^2 n)}$
[Geffert&Mereghetti&P.'03]

Conjecture of Sakoda and Sipser (1978):
the costs of 1NFA → 2DFA and 2NFA → 2DFA
in the general case are exponential

Unary Two-Way Automata

Two-Way Automata: Few Technical Details



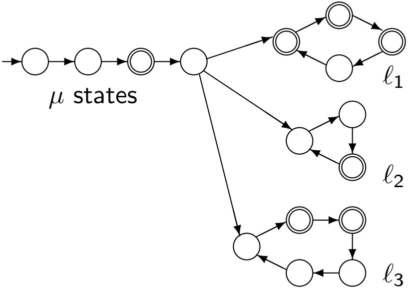
- Input surrounded by the end-markers $\triangleright$ and $\triangleleft$
- $w \in \Sigma^*$ is accepted iff there is a computation
  - with input tape $\triangleright w \triangleleft$
  - starting with the head on $\triangleright$ in the initial state
  - reaching a final state (with the head on $\triangleright$)

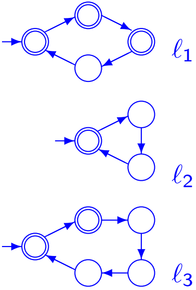
Almost Equivalent Automata

Definition

Two automata A and B are *almost equivalent*
if $L(A)$ and $L(B)$ differ for finitely many strings

Chrobak Normal Form Revisited





Each unary n-state 1NFA A is almost equivalent to a 1NFA B:

- s disjoint loops of lengths $\ell_1, \ldots, \ell_s$, with $\ell_1 + \cdots + \ell_s \leq n$
- at the beginning of the computation,
  B nondeterministically selects a loop $i \in \{1, \ldots, s\}$
- then B counts the input length modulo ℓ_i
- $L(A)$ and $L(B)$ can differ only on strings of length at most $n^2 - n$

A Normal Form for Unary 2NFAs

Theorem ([Geffert&Mereghetti&P.'03])

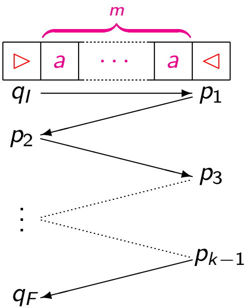
For each unary n-state 2NFA A there exists an almost equivalent 2NFA M s.t.

- *M makes nondeterministic choices and changes the head direction only visiting the end-markers*
- *M has* $N \leq 2n + 2$ *many states*
- $L(A)$ *and* $L(M)$ *can differ only on strings of length* $\leq 5n^2$

A Normal Form for Unary 2NFAs

More details on M:

- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - q_I initial state
  - q_F accepting state
  - Q_i deterministic loop of length ℓ_i
- A computation is a sequence of traversals of the input
- In each traversal M counts the input length modulo one ℓ_i



Remark

If a string is accepted by M then it is accepted by a computation which visits the left end-marker at most $\#Q$ times

Converting Unary 2NFAs into 2DFAs

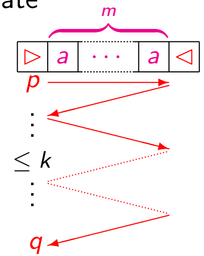
[Geffert&Mereghetti&P.'03]

M unary N-state 2NFA *in normal form*

a^m input string

- For $p, q \in Q$, $k \geq 1$, we consider the predicate
  reachable$(p, q, k) \equiv$
  $\exists$computation path on a^m which
  - starts in the state p on $\triangleright$
  - ends in the state q on $\triangleright$
  - visits $\triangleright$ at most k times

  Then:
  $a^m \in L(M)$ iff *reachable*(q_I, q_F, N) is true



- *reachable*(p, q, k) can be computed by a recursive procedure
- Implemented by a 2DFA with $e^{O(\ln^2 N)}$ states

From Unary 2NFAs to 2DFAs

A	given unary 2NFA	n states
$\Downarrow$		Conversion into normal form
M	almost equivalent 2NFA	$N \le 2n+2$ states
$\Downarrow$		Deterministic simulation
B	2DFA equivalent to M	$e^{O(\ln^2 N)}$ states
$\Downarrow$		Preliminary scan to accept/reject inputs of length $\le 5n^2$ then simulation of B for longer inputs
C	2DFA equivalent to A	$e^{O(\ln^2 n)}$ states

Theorem ([Geffert&Mereghetti&P.'03])

Each unary n-state 2NFA can be simulated by a 2DFA with $e^{O(\ln^2 n)}$ many states

Can this upper bound be reduced to a polynomial?

Upper bound

- superpolynomial
- subexponential

Logspace Classes and Graph Accessibility Problem

L: class of languages accepted in logarithmic space by *deterministic* machines

NL: class of languages accepted in logarithmic space by *nondeterministic* machines

Problem

$L \stackrel{?}{=} NL$

Graph Accessibility Problem GAP

- Given $G = (V, E)$ oriented graph, $s, t \in V$
- Decide whether or not G contains a path from s to t

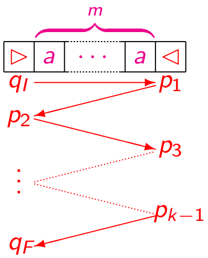
Theorem ([Jones '75])

GAP *is complete for* NL

Hence GAP $\in$ L iff L $=$ NL

Reduction to GAP
[Geffert&P.'11]

M unary 2NFA in normal form, with N states

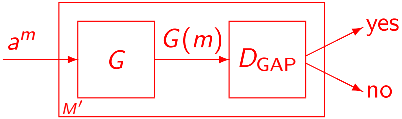


- Accepting computation on a^m
  - sequence of traversals of the input
    - starting in q_I on $\triangleright$
    - ending in q_F on $\triangleright$
- Graph $G(m)$
  - vertices $\equiv$ states
  - edges $\equiv$ traversals on a^m
- a^m is accepted iff $G(m)$ contains a path from q_I to q_F

To decide whether or not $a^m \in L(M)$ reduces to decide GAP for $G(m)$

L = NL $\Rightarrow$ Polynomial Deterministic Simulation!

[Geffert&P.'11]



D_{GAP} logspace bounded *deterministic* machine solving GAP

- $O(\log N)$ space — N=#states of the given 2NFA M
- $poly(N)$ different configurations

$G(m)$ graph associated with a^m

- $O(N^2)$ bits
- $exp(N)$ different configurations — Too many!!!
- bits computed on demand:
  an N-state 1DFA $A_{p,q}$ tests the existence of the edge (p, q)
  trying to simulate a traversal of M from p to q

M' resulting 2DFA — $poly(N)$ many states!!!

From Unary 2NFAs to 2DFAs (under $\mathsf{L} = \mathsf{NL}$)

A	given unary 2NFA	n states
$\Downarrow$		Conversion into normal form
M	almost equivalent 2NFA	$N \leq 2n + 2$ states
$\Downarrow$		Deterministic simulation
B	2DFA equivalent to M	$poly(N)$ states
$\Downarrow$		Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of B for longer inputs
C	2DFA equivalent to A	$poly(n)$ states

Theorem ([Geffert&P.'11])

If $\mathsf{L} = \mathsf{NL}$ then each unary n-state 2NFA can be simulated by a 2DFA with $poly(n)$ many states

Proving the conjecture of Sakoda and Sipser for 2NFA $\rightarrow$ 2DFA in the unary case would separate L and NL in the general case

From Unary 2NFAs to 2DFAs (under L = NL)

A	given unary 2NFA	n states
$\Downarrow$		Conversion into normal form
M	almost equivalent 2NFA	$N \leq 2n+2$ states
$\Downarrow$		Deterministic simulation
B	2DFA equivalent to M	$poly(N)$ states
$\Downarrow$		Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of B for longer inputs
C	2DFA equivalent to A	$poly(n)$ states

Theorem ([Geffert&P.'11])

If L = NL *then each unary n-state 2NFA can be simulated by a 2DFA with poly*(n) *many states*

Theorem ([Kapoutsis&P.'12])

L$/poly \supseteq$ NL *iff each unary n-state 2NFA can be simulated by a 2DFA with poly*(n) *many states*

Normal Form for Unary 2NFAs: Consequences

(i) Subexponential simulation of unary 2NFAs by 2DFAs [Geffert&Mereghetti&P.'03]

(ii) Polynomial simulation of unary 2NFAs by 2DFAs *under the condition* L = NL [Geffert&P.'11]

(iii) Polynomial simulation of unary 2NFAs by unambiguous 2NFAs *(unconditional)* [Geffert&P.'11]

(iv) Polynomial complementation of unary 2NFAs
Inductive counting argument [Geffert&Mereghetti&P.'07]

Normal Form for Unary 2NFAs: Consequences

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Extension to *outer nondeterministic automata*:
[Geffert&Guillon&P.'14]

- general alphabet
- unrestricted head reversals
- nondeterministic choices *only* at the endmarkers

Pushdown Automata and Other Devices

Unary Context-Free Languages

Theorem [Ginsurg&Rice '62]

Each unary context-free languages is regular

How large should be a finite automata equivalent to a given unary context-free grammar or pushdown automaton?

Unary Pushdown Automata

From PDAs of size s, accepting regular languages,
to equivalent 1DFAs

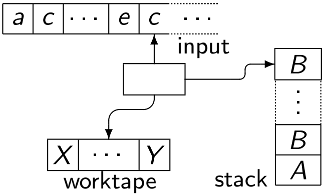
	unary input	general input
PDAs	$2^{poly(s)}$ [P.&Shallit&Wang '02]	non recursive [Meyer&Fischer '71]
deterministic PDAs	$2^{O(s)}$ [P.'09]	$2^{2^{O(s)}}$ [Valiant '75]

All the bounds are tight!

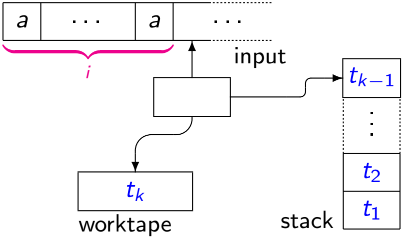
Auxiliary Pushdown Automata (AuxPDAs)

PDAs augmented with an auxiliary worktape

'SPACE' ≡ worktape



1AuxPDAs: How to Count the Input Length

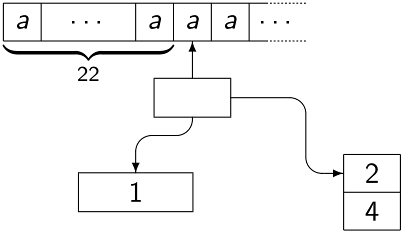


$$i = (1\ 1\ 0\ \cdots\ 1\ 0\ 0\ 0\ 1\ 0)_2 = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_{k-1}} + 2^{t_k}$$

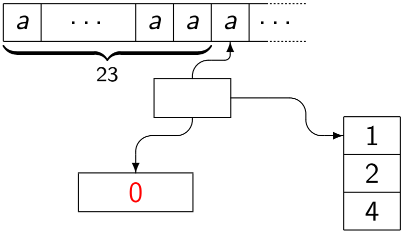
$t_1\ t_2 \quad t_{k-1} \quad t_k$

1AuxPDAs: How to Count the Input Length

$22 = 2^4 + 2^2 + 2^1$

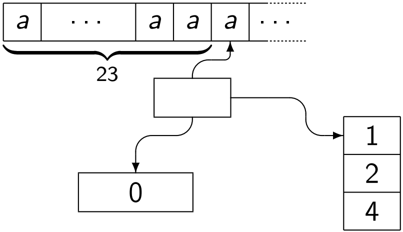


$23 = 2^4 + 2^2 + 2^1 + 2^0$

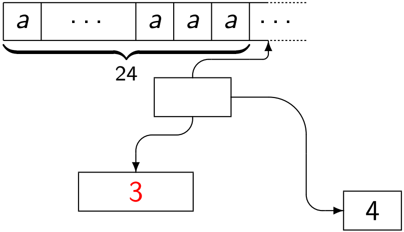


1AuxPDAs: How to Count the Input Length

$$23 = 2^4 + 2^2 + 2^1 + 2^0$$



$$24 = 2^4 + 2^3$$



Example: $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$

- $\mathcal{L}_p$ is nonregular
- $\mathcal{L}_p$ is accepted by a 1AuxPDA M which:
  - scans the input while counting its length
  - accepts iff the pushdown store is empty
    i.e., the binary representation of the input length contains exactly one digit 1
- On input a^n
  the largest integer stored on the worktape is $\lfloor \log_2 n \rfloor$, which is represented in $O(\log \log n)$ space

$$\mathcal{L}_p \in \text{1AuxPDASpace}(\log \log n)$$

Space Bounds on 1AuxPDAs

$\mathcal{L}_p$ is accepted using the *minimum amount of space* for nonregular languages recognition:

Theorem ([P.&Shallit&Wang '02])

If a unary language L is accepted by a 1AuxPDA in $o(\log\log n)$ space then L is regular

In contrast

- with a binary alphabet,
- and space measured on the 'less' expensive accepting computation:

Theorem ([Chytil '86])

For each $k \geq 2$ there is a non context-free language L_k accepted by a 1AuxPDA in $O(\underbrace{\log \ldots \log}_{k} n)$ space

Two-way Pushdown Automata (2PDAs)

- More powerful than PDAs, e.g., $\{a^n b^n c^n \mid n \geq 0\}$
- 2DPDAs can be simulated by RAMs in *linear time* [Cook '71]

Main open problems:

- Power of nondeterminism, i.e., 2DPDAs vs 2PDA
- 2DPDAs vs linear bounded automata

Unary 2PDAs

- Very powerful models, even in the deterministic version

Theorem ([Monien '84])

The unary encoding of each language in P *is accepted by a 2DPDA*

- With a *constant number* of input head reversals they accept only regular languages [Liu&Weiner '68]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$ accepted by a 2DPDA making $\approx \log_2 n$ reversals

Problem

Does there exist a unary nonregular language accepted by a 2PDA making $o(\log n)$ head reversals?

Multi-Head Finite Automata

- More powerful than one-head finite automata, even if the heads are *one-way*, e.g., $\{a^n b^n \mid n \geq 0\}$
- *Unary case:* with a *constant number* of head reversals they accept only regular languages [Sudborough '74]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$ accepted by a 2-head automaton making $\approx \log_2 n$ reversals

Problem

Does there exist a unary nonregular language accepted by a multi-head automaton making $o(\log n)$ head reversals?

- Unary multi-head 2PDAs making $O(1)$ input head reversals accept only regular languages [Ibarra '74]

Conclusion

Final Remarks

Unary Automata and Languages

- Interesting properties and differences with respect to the general case
- Special methods (e.g., from number theory)
- Important relationships with the general case
- Several open problems

Thank you for your attention!