# Investigations on Automata and Languages over a Unary Alphabet

Giovanni Pighizzini

Dipartimento di Informatica
Università degli Studi di Milano, Italy

FLA 2016 – Napoli
January 14-16, 2016

UNIVERSITÀ DEGLI STUDI
DI MILANO

# Unary or Tally Languages

- One letter alphabet $\Sigma = \{a\}$
- Many differences with the general case have been discovered

  First example:

  **Theorem [Ginsurg&Rice '62]**

  *Each unary context-free languages is regular*

- Structural complexity: classes of *tally sets*
  - Hartmanis, 1972
  - Book, 1974, 1979
  - ...

  Space complexity:
  - Alt&Mehlhorn, 1975
  - Geffert, 1993
  - ...

# Unary or Tally Languages

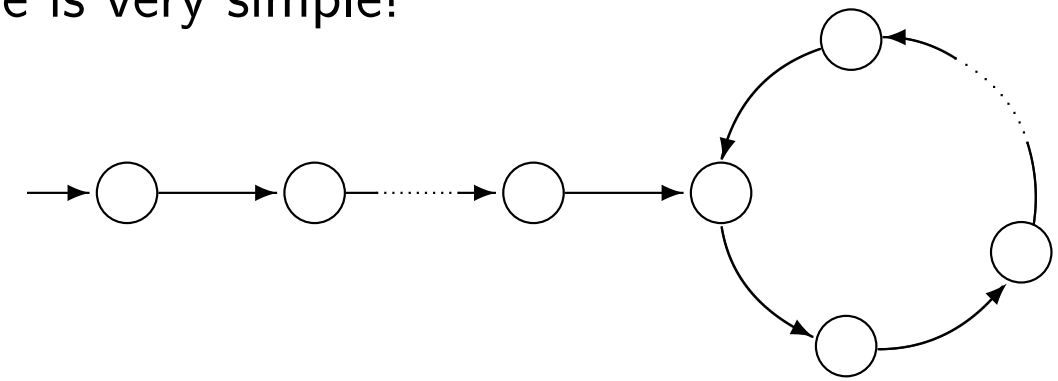
This talk:

- Focus mainly on *descriptional complexity aspects*
  - Optimal simulations between variants of unary automata
  - Unary two-way automata:
    connection with the question $\mathrm{L} \stackrel{?}{=} \mathrm{NL}$
  - Unary context-free grammars and pushdown automata
- Devices accepting nonregular languages

# Unary Automata

# Unary One-Way Deterministic Automata (1DFAs)

The structure is very simple!



**Theorem**

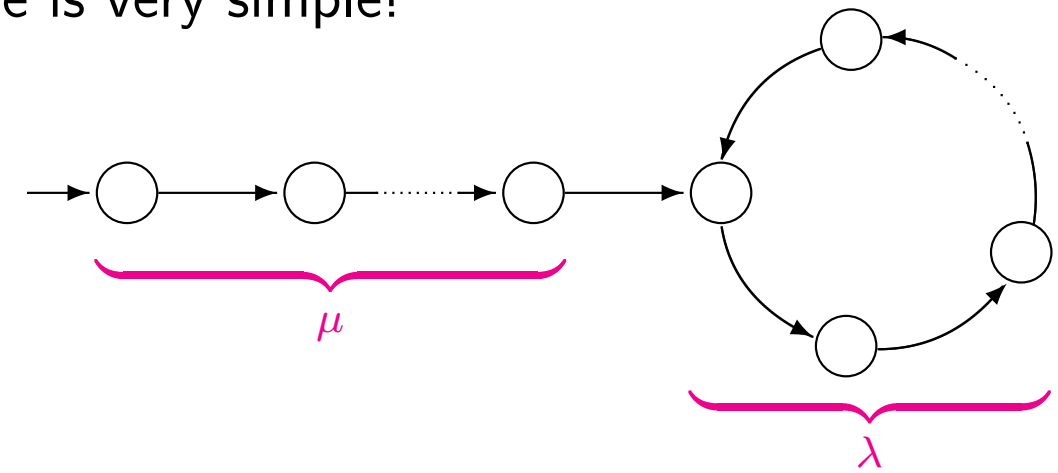
*$L \subseteq \{a\}^*$ is regular iff $\exists \mu \geq 0, \lambda \geq 1$ s.t.*

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L$$

When $\mu = 0$ the language $L$ is said to be *cyclic*

# Unary One-Way Deterministic Automata (1DFAs)

The structure is very simple!




## Theorem

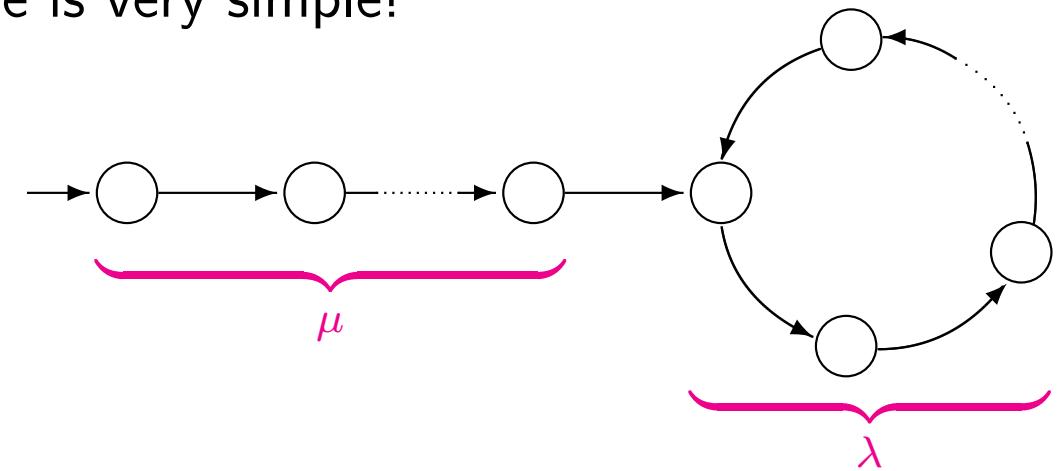
*$L \subseteq \{a\}^*$ is regular iff $\exists \mu \geq 0, \lambda \geq 1$ s.t.*

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L$$

When $\mu = 0$ the language $L$ is said to be *cyclic*

# Unary One-Way Deterministic Automata (1DFAs)

The structure is very simple!




## Theorem

$L \subseteq \{a\}^*$ *is regular iff* $\exists \mu \geq 0, \lambda \geq 1$ *s.t.*

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L$$

When $\mu = 0$ the language $L$ is said to be *cyclic*

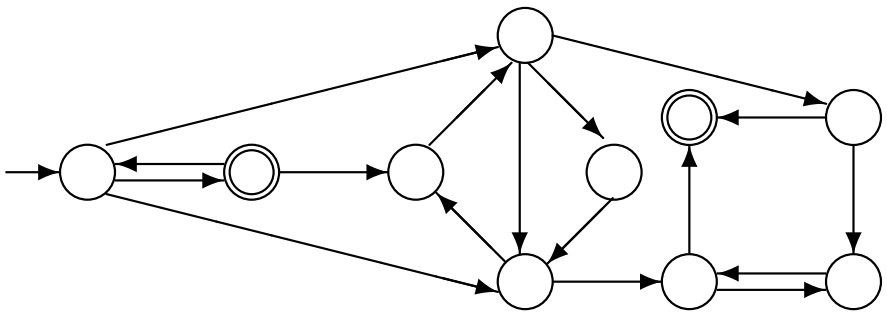
# Unary One-Way Nondeterministic Automata (1NFAs)

The structure can be very complicate!

*Each direct graph with*

- *a vertex selected as initial state*
- *some vertices selected as final states*

*is the transition diagram of a unary 1NFA!*



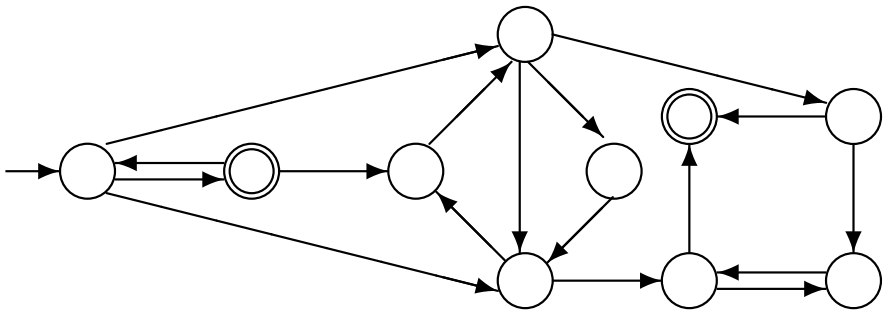
# Unary One-Way Nondeterministic Automata (1NFAs)

The structure can be very complicate!

*Each direct graph with*

- *a vertex selected as initial state*
- *some vertices selected as final states*

*is the transition diagram of a unary 1NFA!*

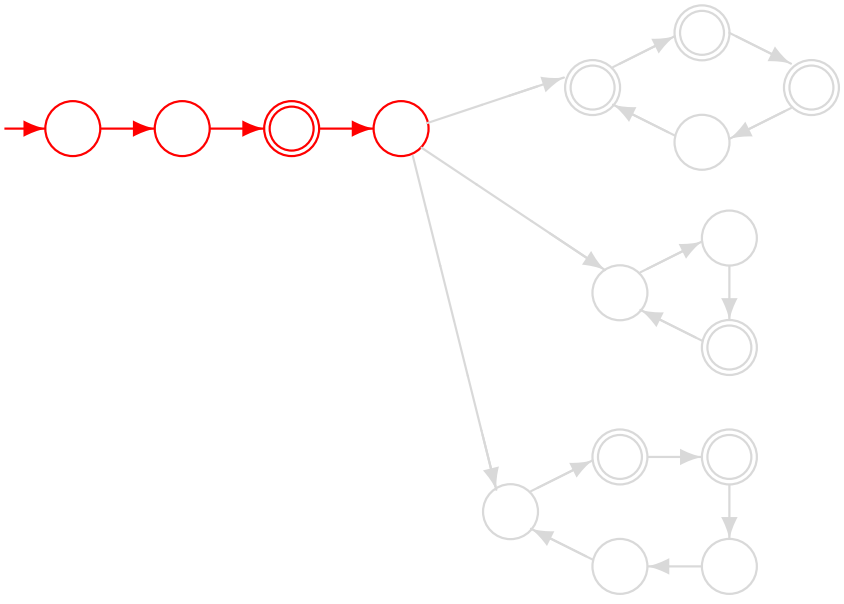


However, we can always obtain an equivalent 1NFA with a

- *simple* and
- *not too big*

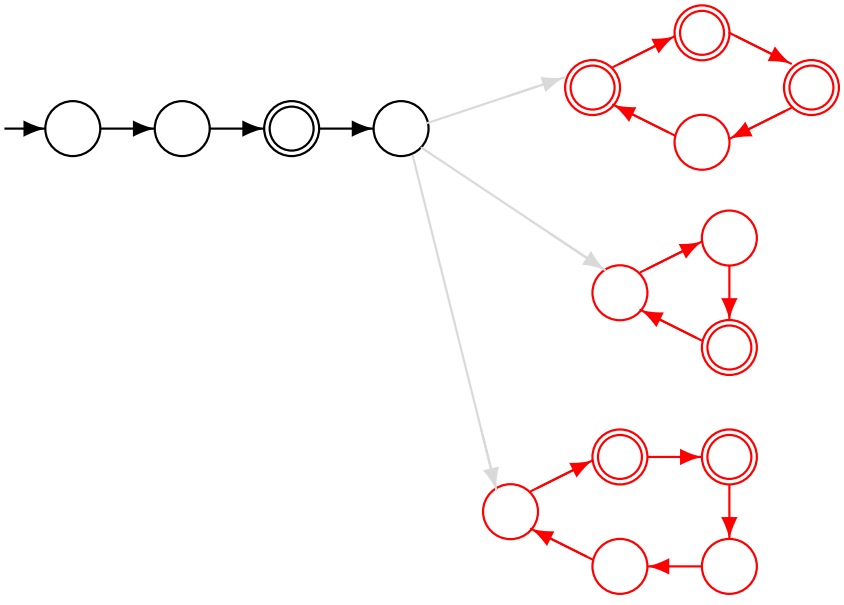
transition graph

# Chrobak Normal Form for 1NFAs



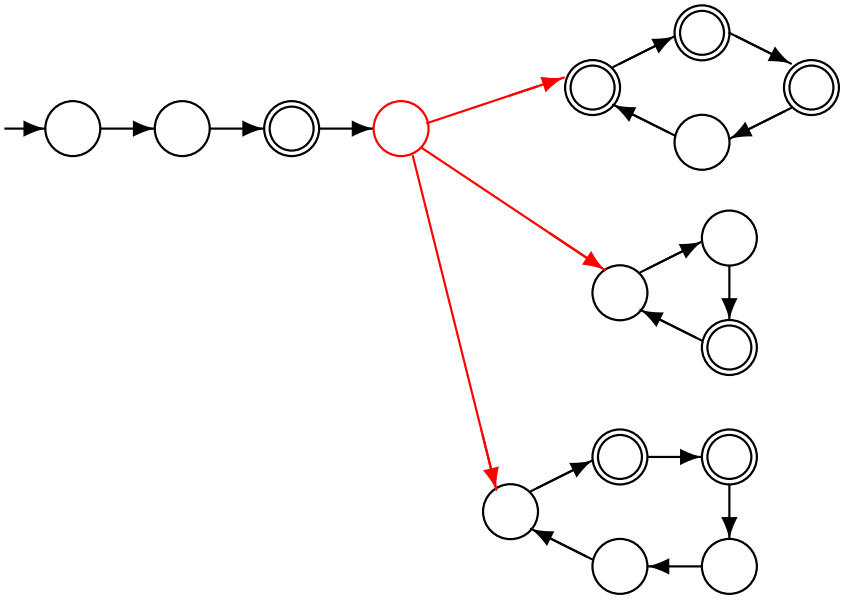
- An initial *deterministic path*
- Some disjoint *deterministic loops*
- *Only one nondeterministic decision*

# Chrobak Normal Form for 1NFAs



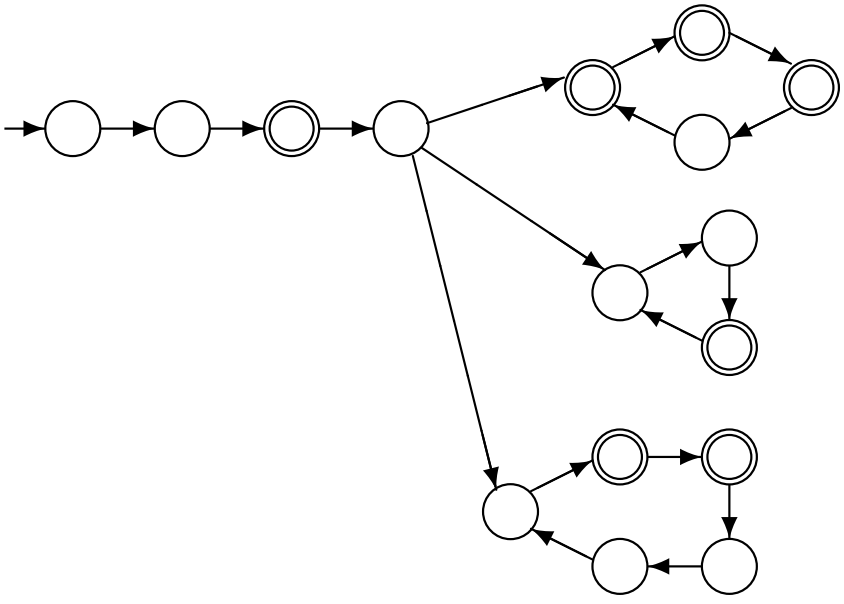
- An initial *deterministic path*
- Some disjoint *deterministic loops*
- *Only one nondeterministic decision*

# Chrobak Normal Form for 1NFAs



- An initial *deterministic path*
- Some disjoint *deterministic loops*
- *Only one nondeterministic decision*

# Chrobak Normal Form for 1NFAs



- An initial *deterministic path*
- Some disjoint *deterministic loops*
- *Only one nondeterministic decision*

**Theorem ([Chrobak '86])**

*Each unary n-state 1NFA can be converted into an equivalent 1NFA in Chrobak normal form with*

- *an initial path of* $O(n^2)$ *states*
- *total number of states in the loops* $\leq n$

# Conversion to Chrobak Normal Form for 1NFAs

- Subtle error in the original proof fixed by To (2009)
- Different transformation proposed by Geffert (2007)
- Polynomial time conversion algorithms
  by Martinez (2004), Gawrychowski (2011), Sawa (2013)
- From the results by Geffert and Gawrychowski:
  - length of the initial path $\leq n^2 - n$
  - total number of states in the loops $\leq n - 1$
    (except when the given 1NFA is the trivial loop of $n$ states)

# Conversion to Chrobak Normal Form for 1NFAs

- Subtle error in the original proof fixed by To (2009)
- Different transformation proposed by Geffert (2007)
- Polynomial time conversion algorithms
  by Martinez (2004), Gawrychowski (2011), Sawa (2013)
- From the results by Geffert and Gawrychowski:
  - length of the initial path $\leq n^2 - n$
  - total number of states in the loops $\leq n - 1$
    (except when the given 1NFA is the trivial loop of $n$ states)

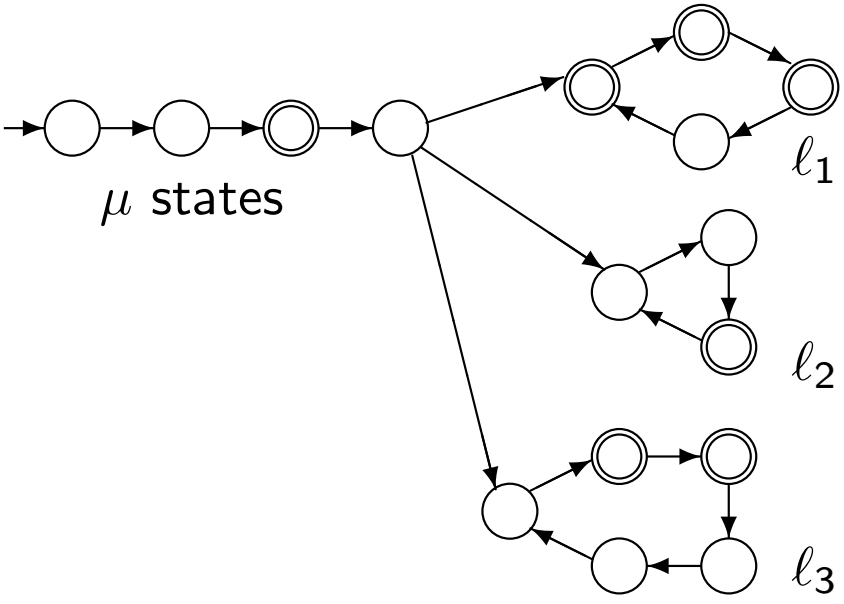
# Conversion to Chrobak Normal Form for 1NFAs

- Subtle error in the original proof fixed by To (2009)
- Different transformation proposed by Geffert (2007)
- Polynomial time conversion algorithms
  by Martinez (2004), Gawrychowski (2011), Sawa (2013)
- From the results by Geffert and Gawrychowski:
  - length of the initial path $\leq n^2 - n$
  - total number of states in the loops $\leq n - 1$
    (except when the given 1NFA is the trivial loop of $n$ states)

# Conversion to Chrobak Normal Form for 1NFAs

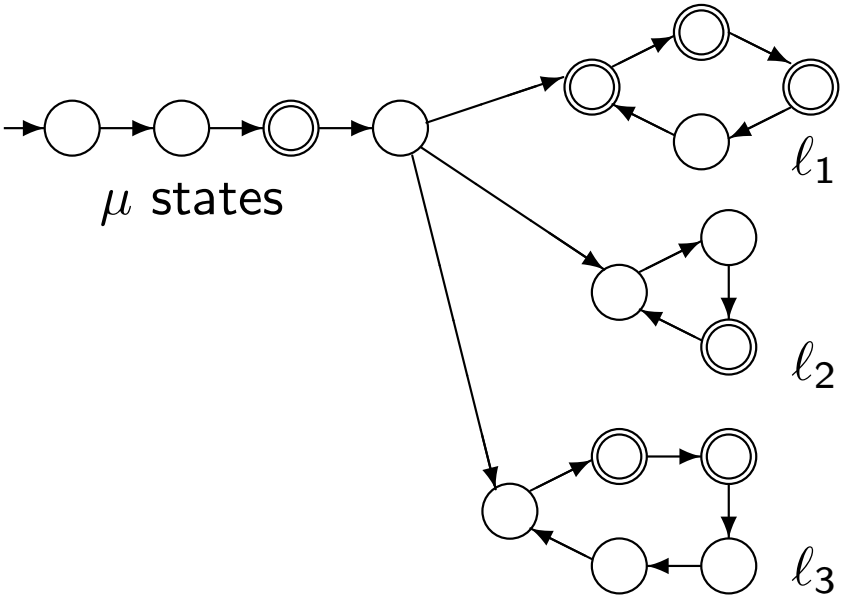
- Subtle error in the original proof fixed by To (2009)
- Different transformation proposed by Geffert (2007)
- Polynomial time conversion algorithms
  by Martinez (2004), Gawrychowski (2011), Sawa (2013)
- From the results by Geffert and Gawrychowski:
  - length of the initial path $\leq n^2 - n$
  - total number of states in the loops $\leq n - 1$
    (except when the given 1NFA is the trivial loop of $n$ states)

# Removing Nondeterminism from Unary Automata



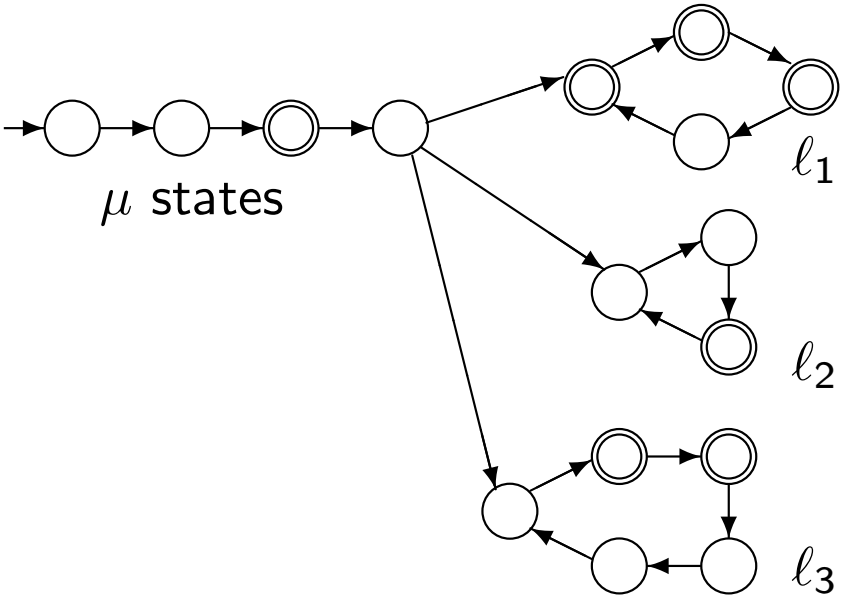

- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

# Removing Nondeterminism from Unary Automata



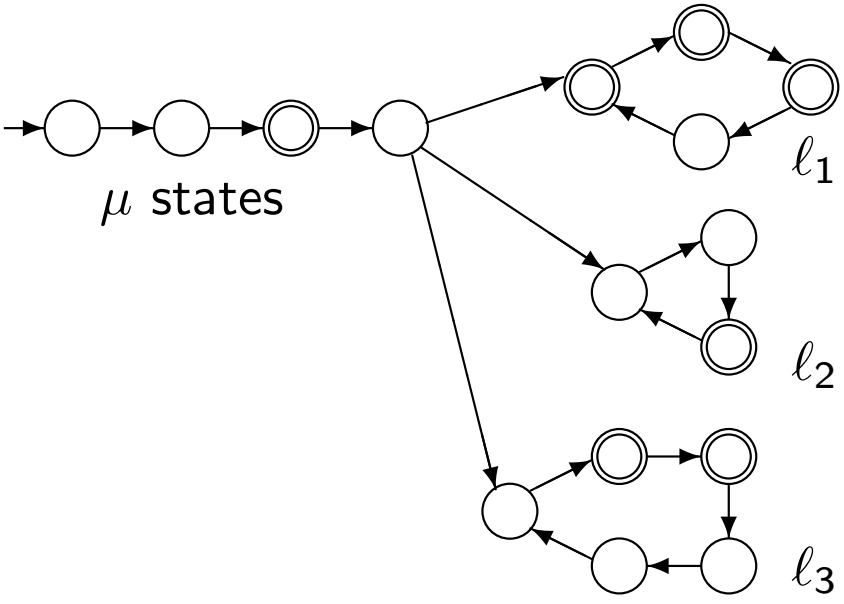

- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

# Removing Nondeterminism from Unary Automata



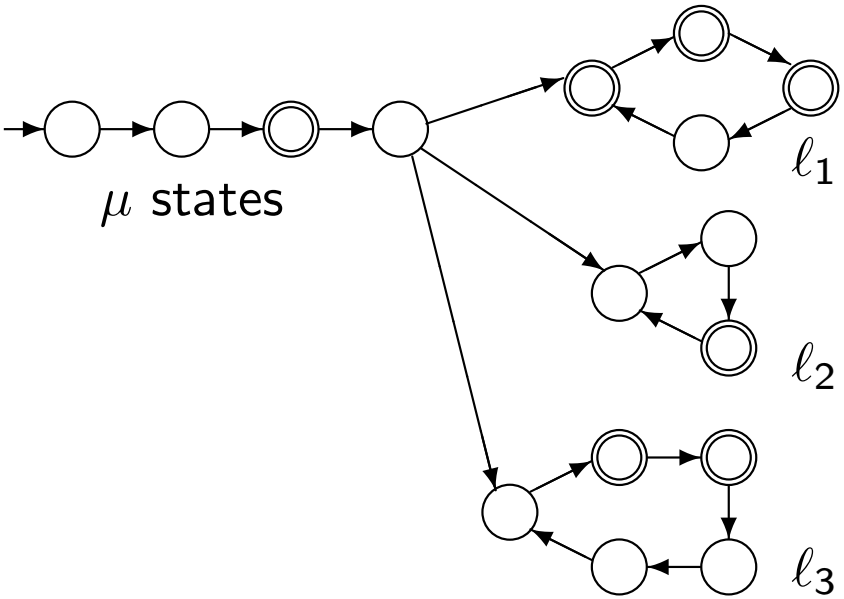

- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

# Removing Nondeterminism from Unary Automata




- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

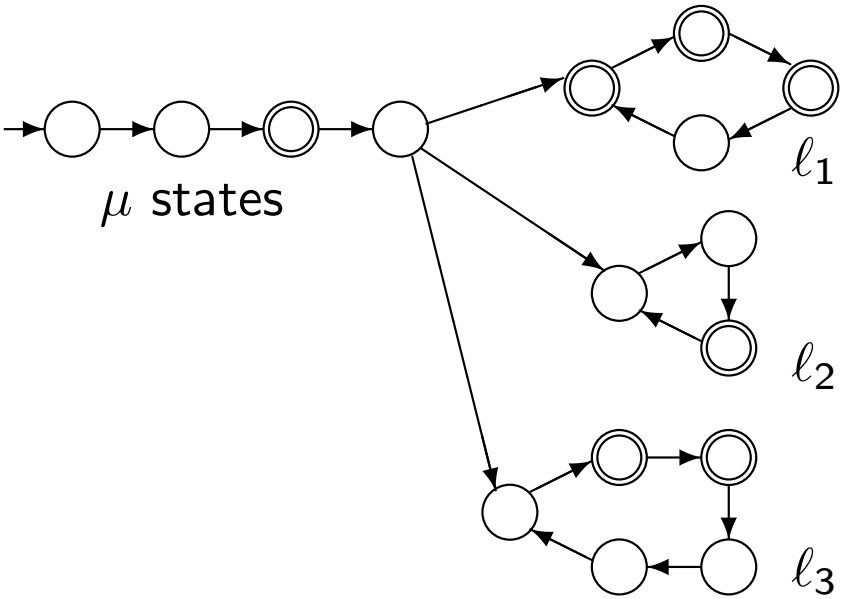
# Removing Nondeterminism from Unary Automata




- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

How large can be $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$?

# Removing Nondeterminism from Unary Automata

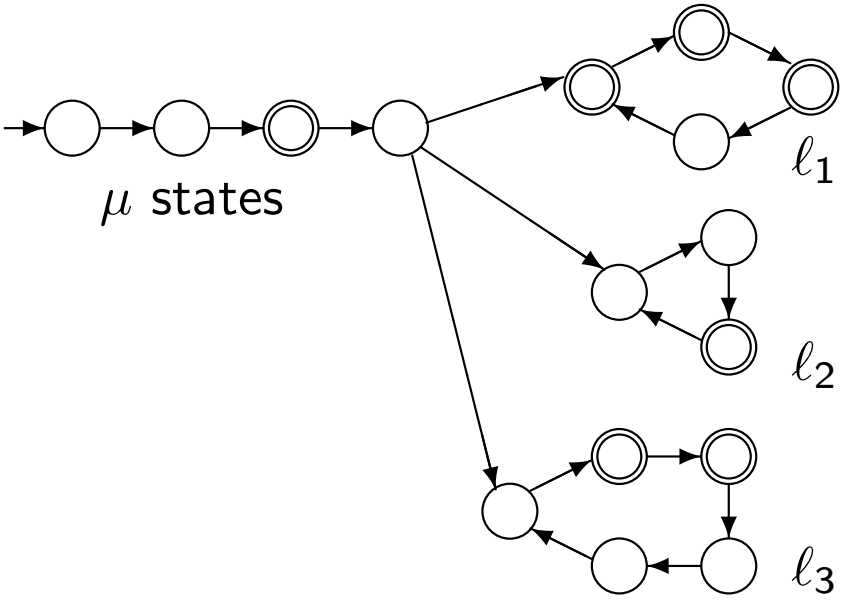



- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

How large can be $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$?

$$F(n) = \max\{\mathrm{lcm}\{\ell_1, \ell_2, \ldots, \ell_s\} \mid s \geq 1 \wedge \ell_1 + \ell_2 + \cdots + \ell_s \leq n\}$$

# Removing Nondeterminism from Unary Automata



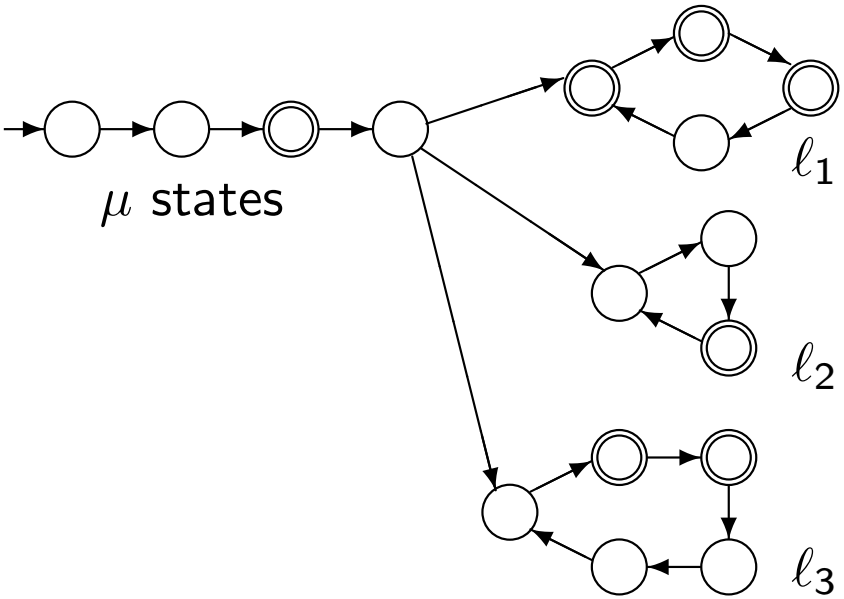

- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

How large can be $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$?

$$F(n) = \max\{\mathrm{lcm}\{\ell_1, \ell_2, \ldots, \ell_s\} \mid s \geq 1 \wedge \ell_1 + \ell_2 + \cdots + \ell_s \leq n\}$$

Landau's function (1903) $F(n) = \mathrm{e}^{\Theta(\sqrt{n \ln n})}$ [Szalay '80]

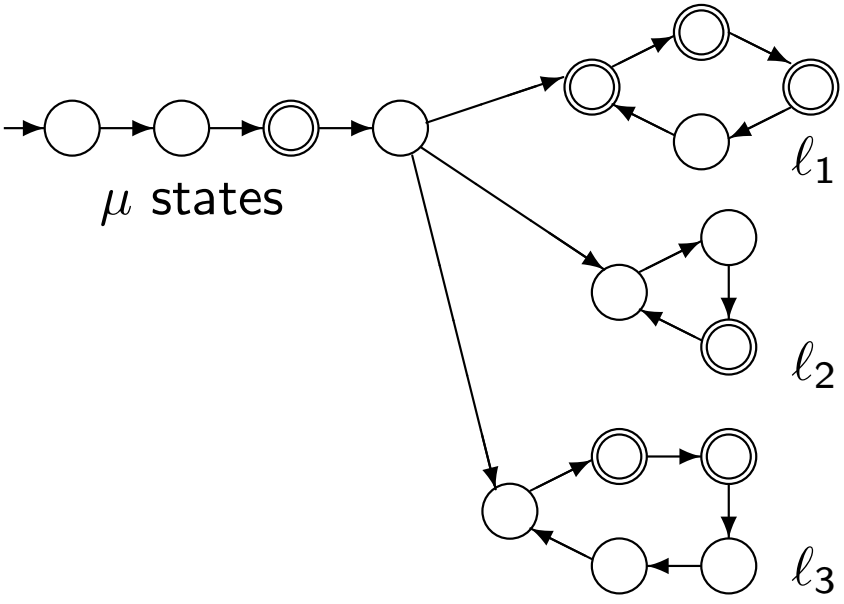
# Removing Nondeterminism from Unary Automata




- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

- $F(n)$ states are also necessary in the worst case [Chrobak '86]

# Removing Nondeterminism from Unary Automata



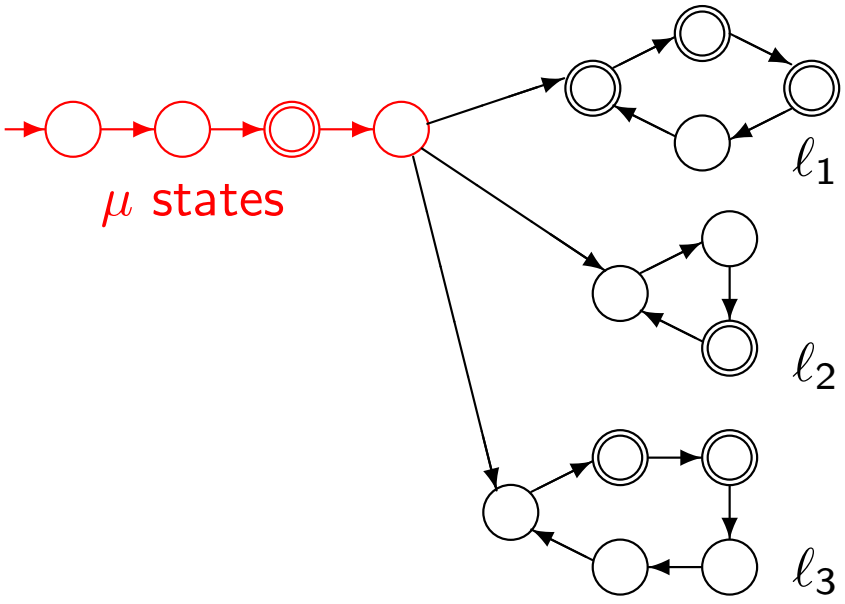

- Keep the same initial path
- Simulate all the loops "in parallel"
- A loop of $\mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$ many states is enough
- Total number of states $\leq \mu + \mathrm{lcm}\{\ell_1, \ell_2, \ldots\}$
- From a $n$-state 1NFA: $\mu = O(n^2)$, $\ell_1 + \ell_2 + \cdots \leq n$

- $F(n)$ states are also necessary in the worst case [Chrobak '86]

**Theorem ([Ljubič '64, Chrobak '86])**

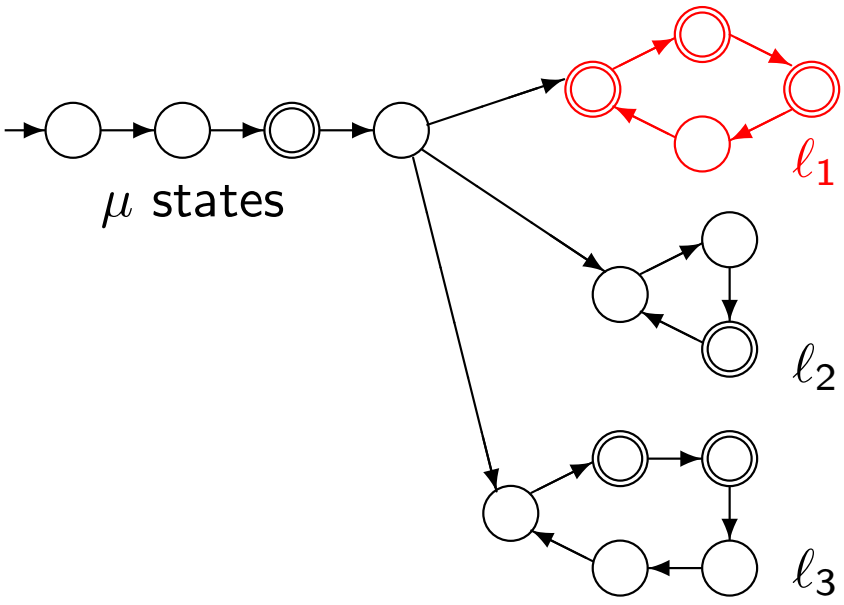
*The state cost of the simulation of unary n-state 1NFAs by equivalent 1DFAs is* $\mathrm{e}^{\Theta(\sqrt{n \ln n})}$

# From Chrobak Normal Form to Two-Way Automata



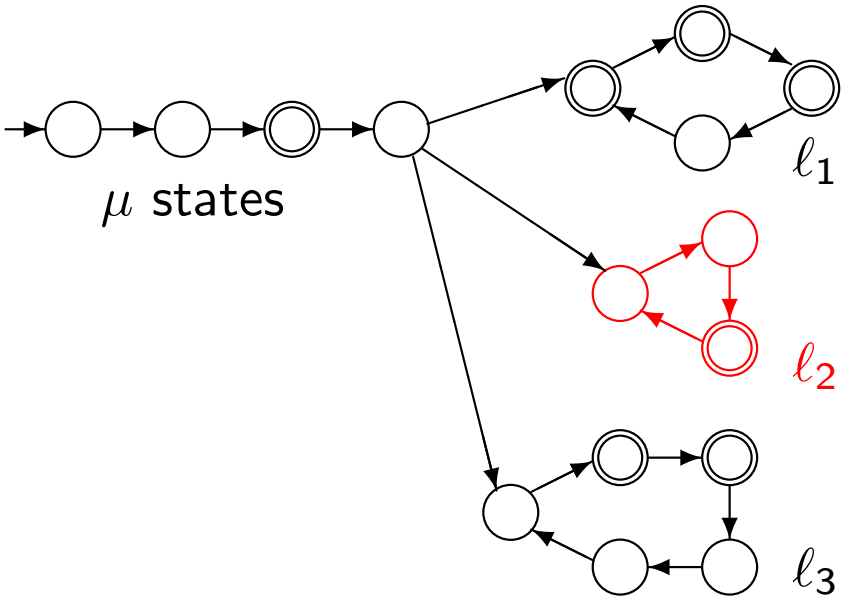

- Check if the input is "short" and accepted on the initial path $\mu + 1$ states
- Check if the input is accepted on the first loop $\ell_1$ states
- Check if the input is accepted on the second loop $\ell_2$ states
- Check if the input is accepted on the third loop $\ell_3$ states

# From Chrobak Normal Form to Two-Way Automata



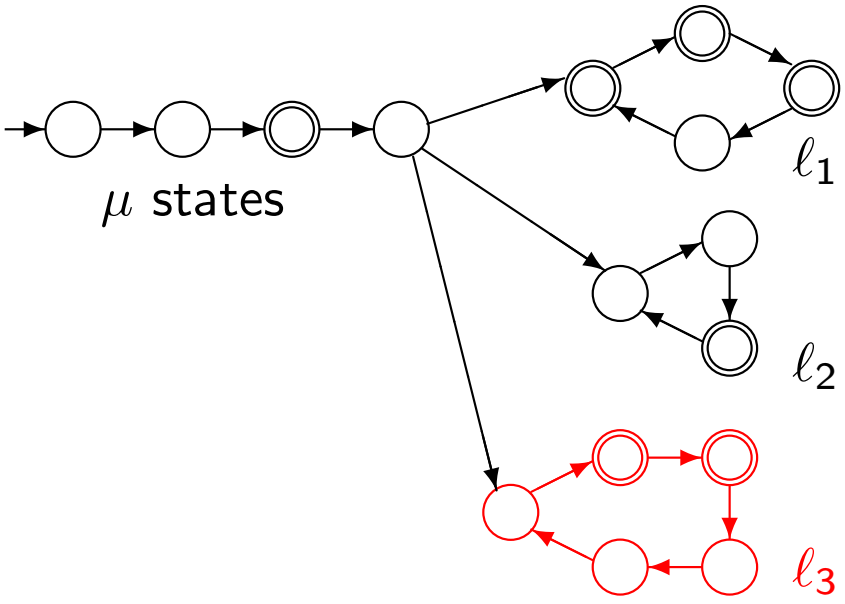

- Check if the input is "short" and accepted on the initial path $\mu + 1$ states
- Check if the input is accepted on the first loop $\ell_1$ states
- Check if the input is accepted on the second loop $\ell_2$ states
- Check if the input is accepted on the third loop $\ell_3$ states

# From Chrobak Normal Form to Two-Way Automata




- Check if the input is "short" and accepted on the initial path $\mu + 1$ states
- Check if the input is accepted on the first loop $\ell_1$ states
- Check if the input is accepted on the second loop $\ell_2$ states
- Check if the input is accepted on the third loop $\ell_3$ states

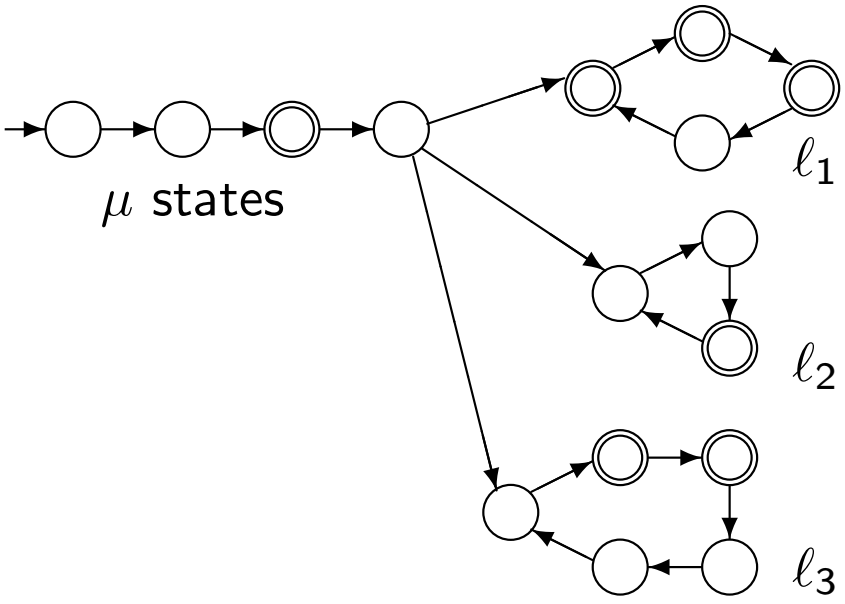
# From Chrobak Normal Form to Two-Way Automata




- Check if the input is "short" and accepted on the initial path $\mu + 1$ states
- Check if the input is accepted on the first loop $\ell_1$ states
- Check if the input is accepted on the second loop $\ell_2$ states
- Check if the input is accepted on the third loop $\ell_3$ states

$\mu + \ell_1 + \ell_2 + \cdots + 2$ states are sufficient!

# From Chrobak Normal Form to Two-Way Automata

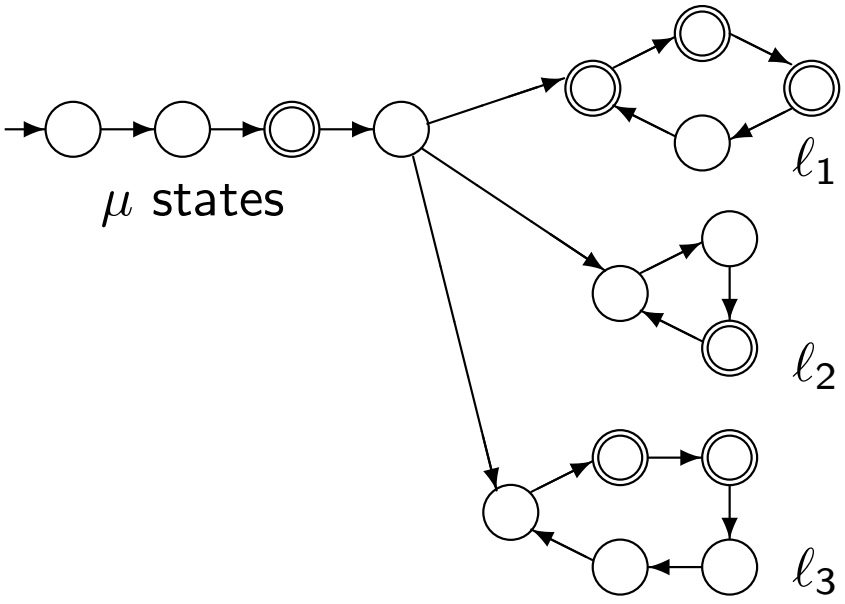



- Check if the input is "short" and accepted on the initial path $\mu + 1$ states
- Check if the input is accepted on the first loop $\ell_1$ states
- Check if the input is accepted on the second loop $\ell_2$ states
- Check if the input is accepted on the third loop $\ell_3$ states

$\mu + \ell_1 + \ell_2 + \cdots + 2$ states are sufficient!

This number is also necessary in the worst case [Chrobak '86]

# From Chrobak Normal Form to Two-Way Automata




- Check if the input is "short" and accepted on the initial path $\mu + 1$ states
- Check if the input is accepted on the first loop $\ell_1$ states
- Check if the input is accepted on the second loop $\ell_2$ states
- Check if the input is accepted on the third loop $\ell_3$ states

$\mu + \ell_1 + \ell_2 + \cdots + 2$ states are sufficient!

This number is also necessary in the worst case [Chrobak '86]

**Theorem**

*The state cost of the simulation of unary n-state 1NFAs by 2DFAs is $\Theta(n^2)$*

# Optimal Simulations Between Unary Automata

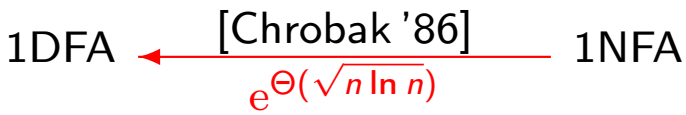
1DFA

1NFA

2DFA

2NFA

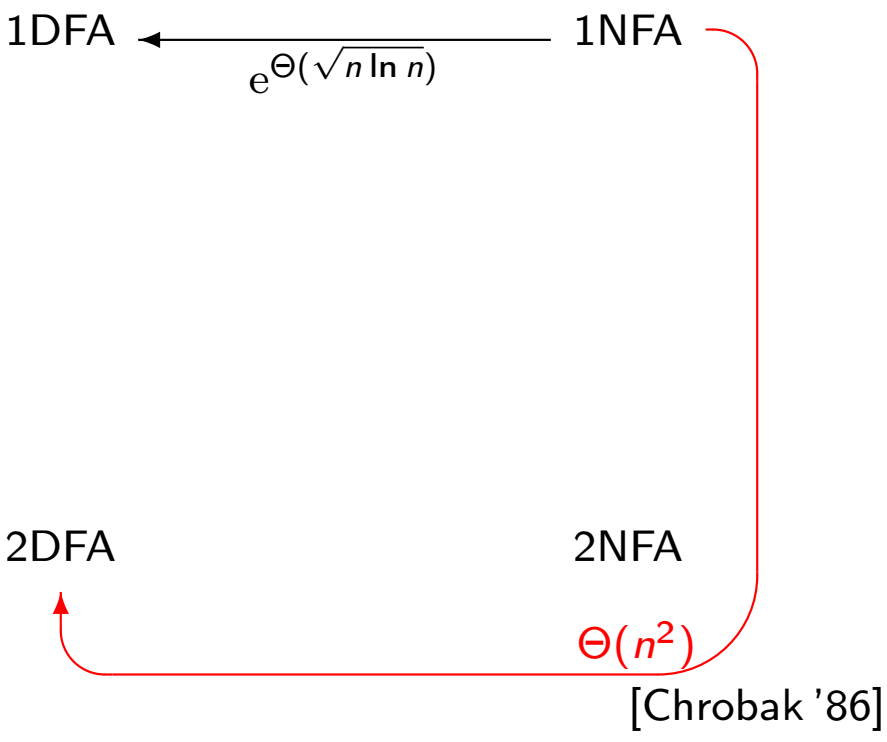
# Optimal Simulations Between Unary Automata



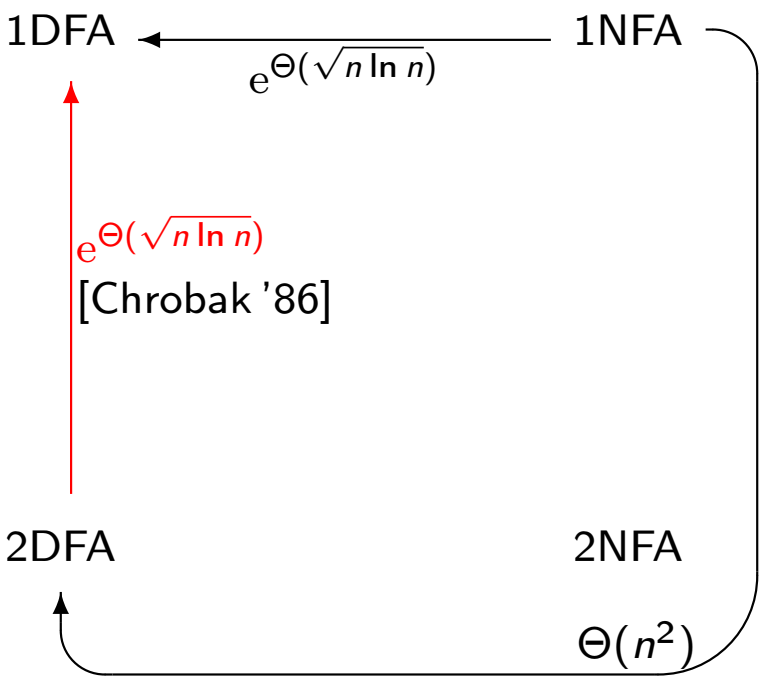

2DFA

2NFA

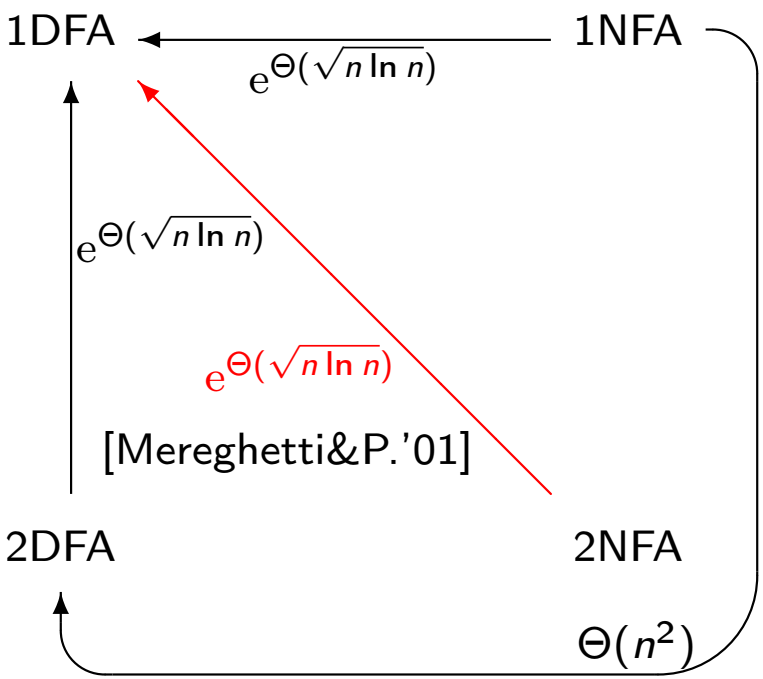
# Optimal Simulations Between Unary Automata



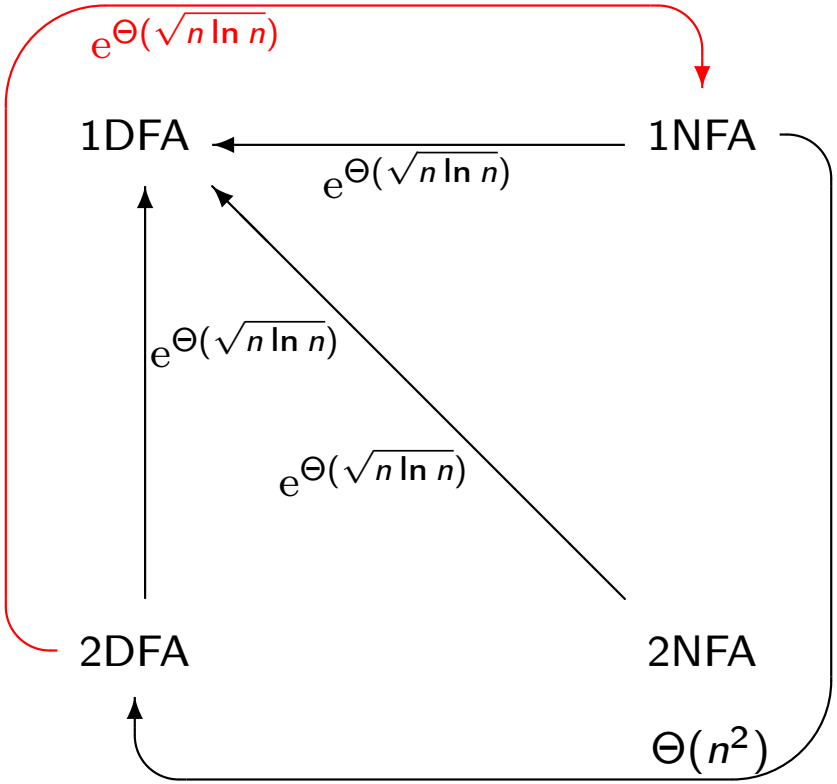
# Optimal Simulations Between Unary Automata



# Optimal Simulations Between Unary Automata

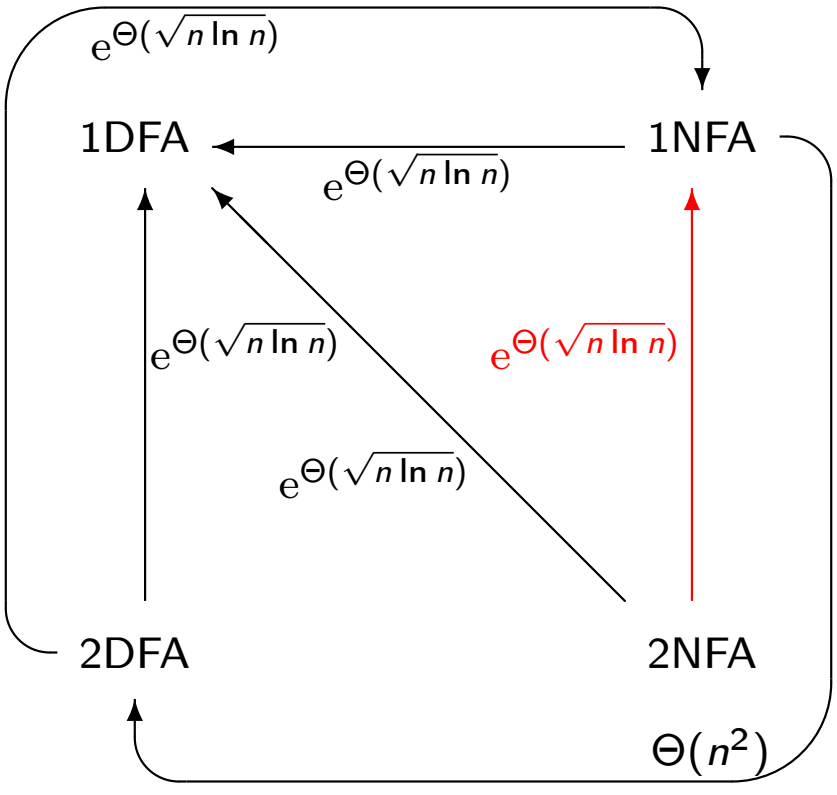


# Optimal Simulations Between Unary Automata



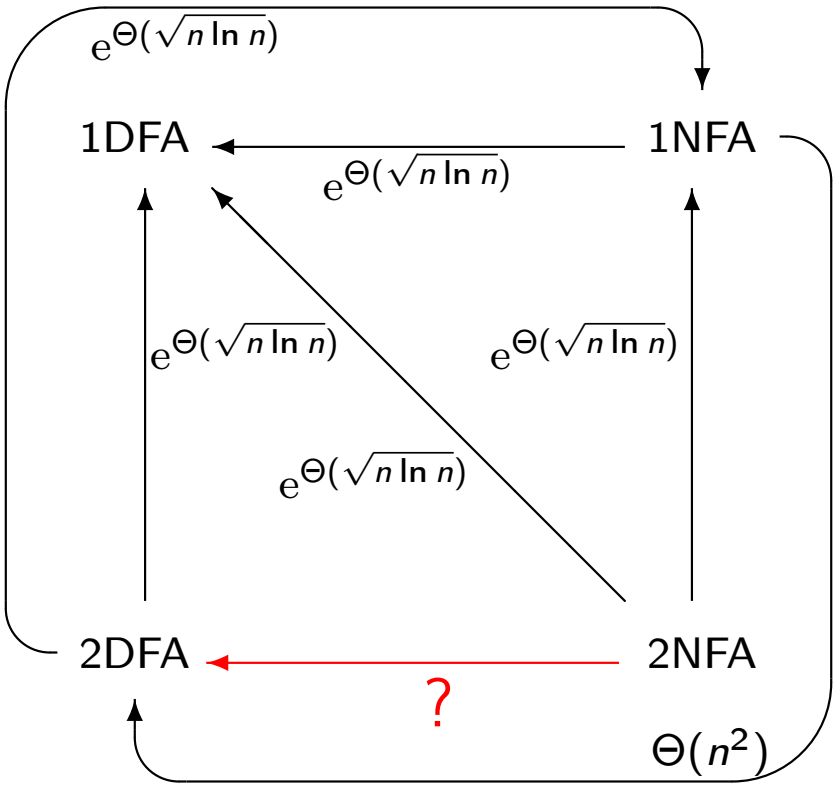

from 2DFA $\rightarrow$ 1DFA

# Optimal Simulations Between Unary Automata




from 2NFA $\rightarrow$ 1DFA

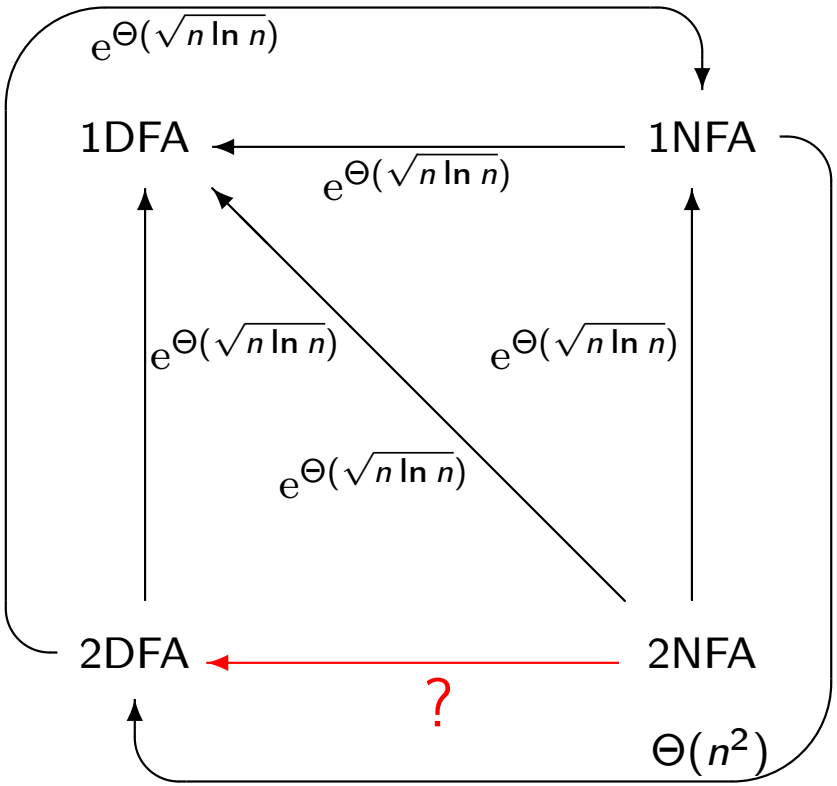
# Optimal Simulations Between Unary Automata




2NFA $\rightarrow$ 2DFA Open!

- upper bound $e^{\Theta(\sqrt{n \ln n})}$ (from 2NFA $\rightarrow$ 1DFA)
- lower bound $\Omega(n^2)$ (from 1NFA $\rightarrow$ 2DFA)

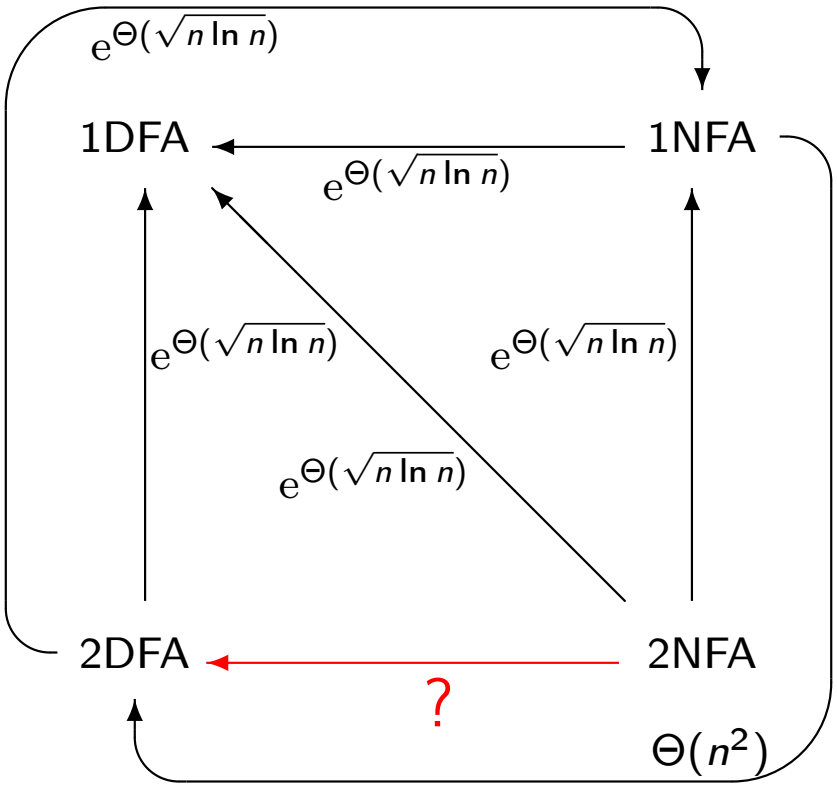
# Optimal Simulations Between Unary Automata




2NFA $\rightarrow$ 2DFA Open!

- upper bound $e^{\Theta(\sqrt{n \ln n})}$ (from 2NFA $\rightarrow$ 1DFA)
- lower bound $\Omega(n^2)$ (from 1NFA $\rightarrow$ 2DFA)

# Optimal Simulations Between Unary Automata

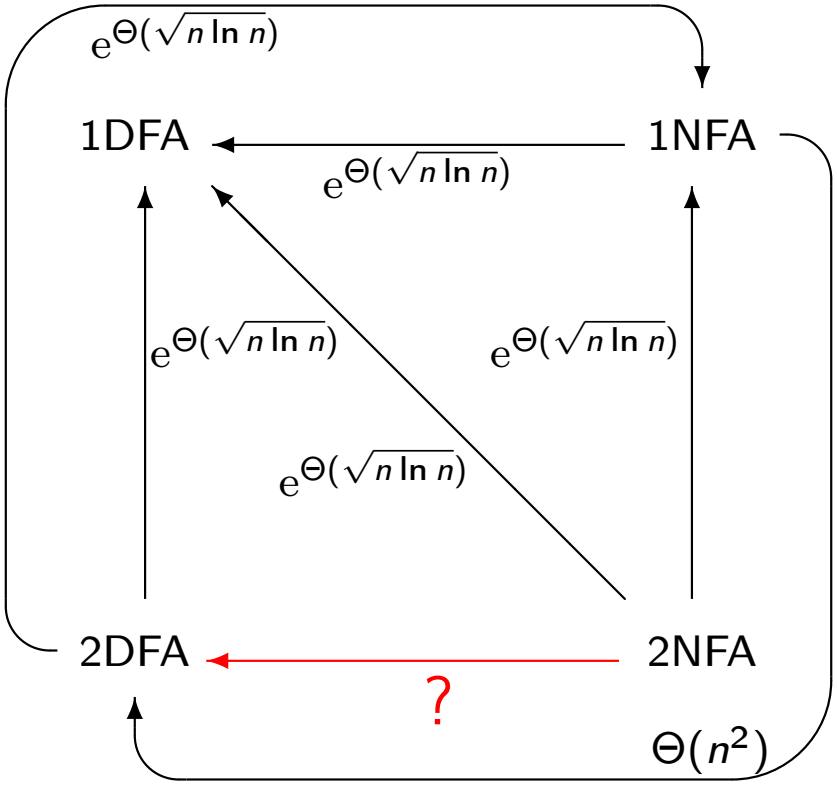



2NFA $\to$ 2DFA Open!

- upper bound $e^{\Theta(\sqrt{n \ln n})}$ (from 2NFA $\to$ 1DFA)
- lower bound $\Omega(n^2)$ (from 1NFA $\to$ 2DFA)

Better upper bound $e^{O(\ln^2 n)}$ [Geffert&Mereghetti&P.'03]

# Optimal Simulations Between Unary Automata




2NFA $\to$ 2DFA Open!

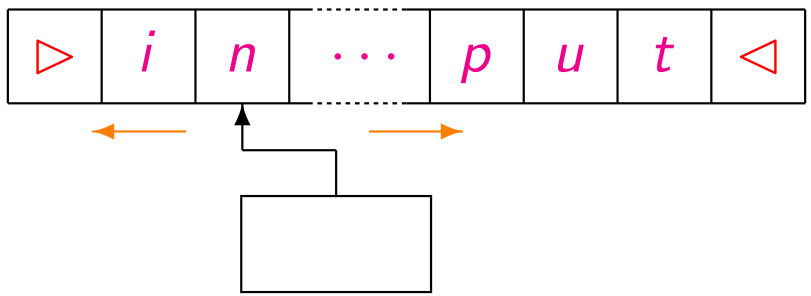
- upper bound $e^{\Theta(\sqrt{n \ln n})}$ (from 2NFA $\to$ 1DFA)
- lower bound $\Omega(n^2)$ (from 1NFA $\to$ 2DFA)

Better upper bound $e^{O(\ln^2 n)}$ [Geffert&Mereghetti&P.'03]

Conjecture of Sakoda and Sipser (1978): the costs of 1NFA $\to$ 2DFA and 2NFA $\to$ 2DFA in the general case are not polymonial

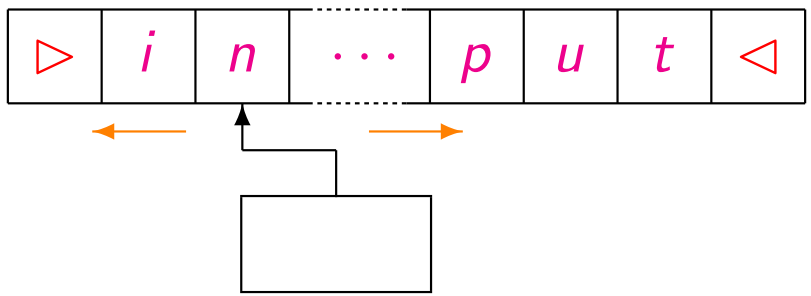
# Unary Two-Way Automata

# Two-Way Automata: Few Technical Details




- Input surrounded by the end-markers $\triangleright$ and $\triangleleft$
- $w \in \Sigma^*$ is accepted iff there is a computation
  - with input tape $\triangleright w \triangleleft$
  - starting with the head on $\triangleright$ in the initial state
  - reaching a final state (with the head on $\triangleright$)

# Two-Way Automata: Few Technical Details



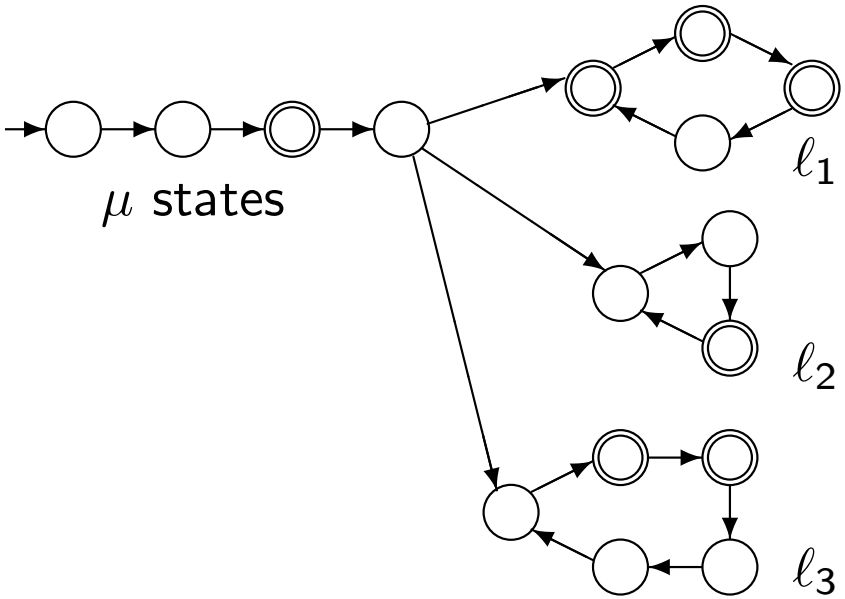

- Input surrounded by the end-markers ▷ and ◁
- $w \in \Sigma^*$ is accepted iff there is a computation
  - with input tape $\triangleright w \triangleleft$
  - starting with the head on ▷ in the initial state
  - reaching a final state (with the head on ▷)

# Almost Equivalent Automata

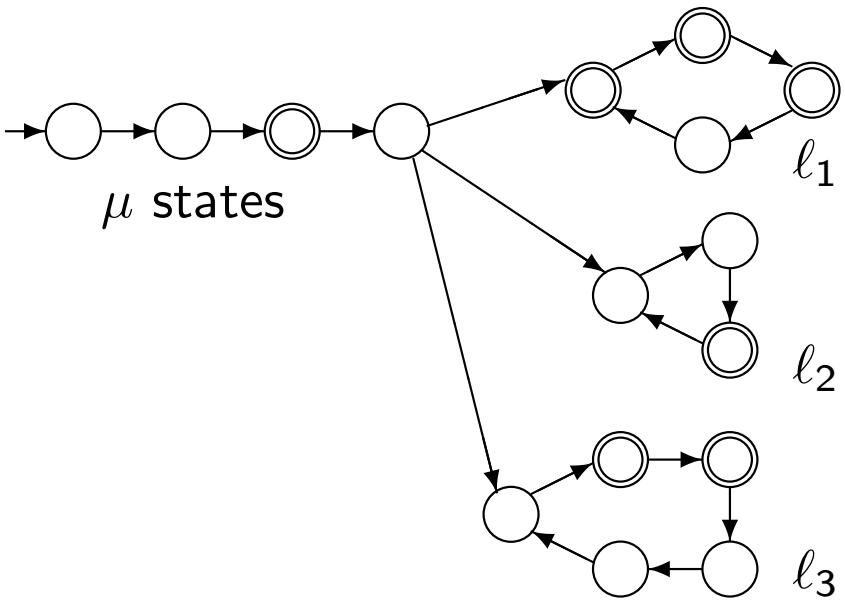
**Definition**

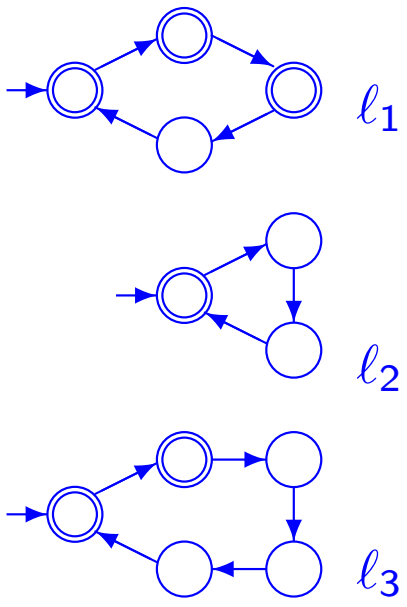
Two automata $A$ and $B$ are *almost equivalent* if $L(A)$ and $L(B)$ differ for finitely many strings

# Chrobak Normal Form Revisited

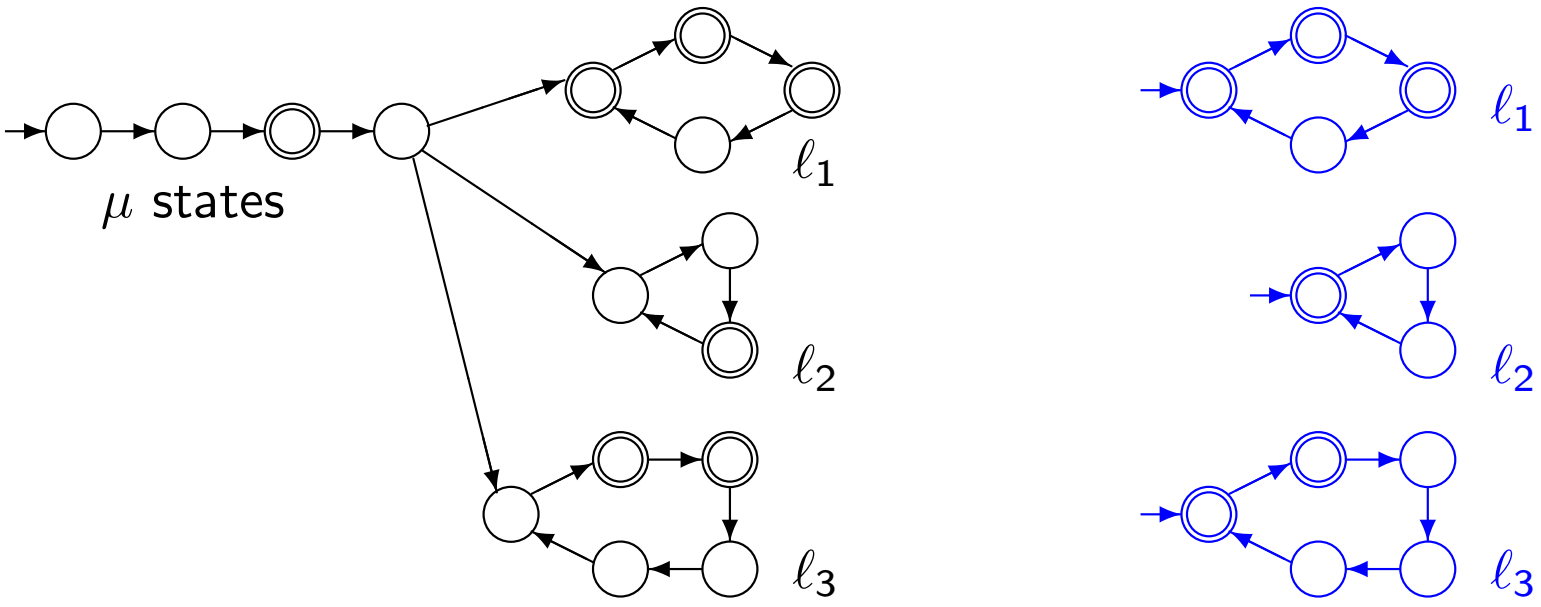


# Chrobak Normal Form Revisited



# Chrobak Normal Form Revisited




Each unary $n$-state 1NFA $A$ is almost equivalent to a 1NFA $B$:

- $s$ disjoint loops of lengths $\ell_1, \ldots, \ell_s$, with $\ell_1 + \cdots + \ell_s \leq n$
- at the beginning of the computation,
  $B$ nondeterministically selects a loop $i \in \{1, \ldots, s\}$
- then $B$ counts the input length modulo $\ell_i$
- $L(A)$ and $L(B)$ can differ only on strings of length at most $n^2 - n$

# A Normal Form for Unary 2NFAs

**Theorem ([Geffert&Mereghetti&P.'03])**

*For each unary n-state 2NFA $A$ there exists an almost equivalent 2NFA $M$ s.t.*

- *$M$ makes nondeterministic choices and changes the head direction only visiting the end-markers*
- *$M$ has $N \leq 2n+2$ many states*
- *$L(A)$ and $L(M)$ can differ only on strings of length $\leq 5n^2$*

# A Normal Form for Unary 2NFAs

More details on $M$:

- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$

# A Normal Form for Unary 2NFAs

More details on $M$:

- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$

# A Normal Form for Unary 2NFAs

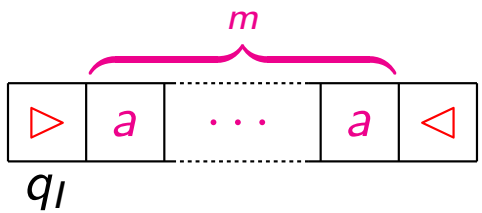
More details on $M$:

- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$

# A Normal Form for Unary 2NFAs

More details on $M$:

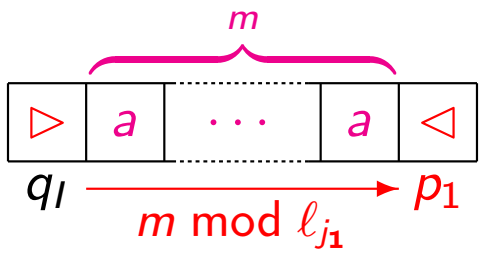
- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$



# A Normal Form for Unary 2NFAs

More details on $M$:

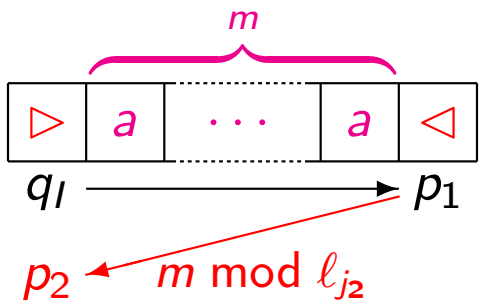
- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$



# A Normal Form for Unary 2NFAs

More details on $M$:

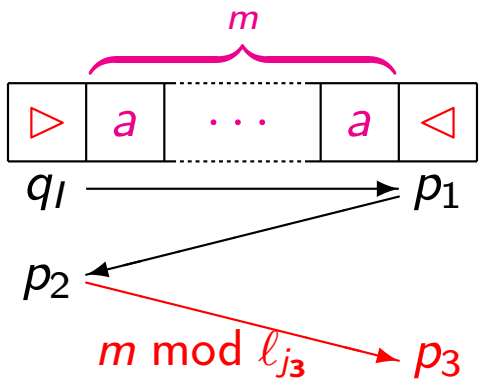
- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$



# A Normal Form for Unary 2NFAs

More details on $M$:

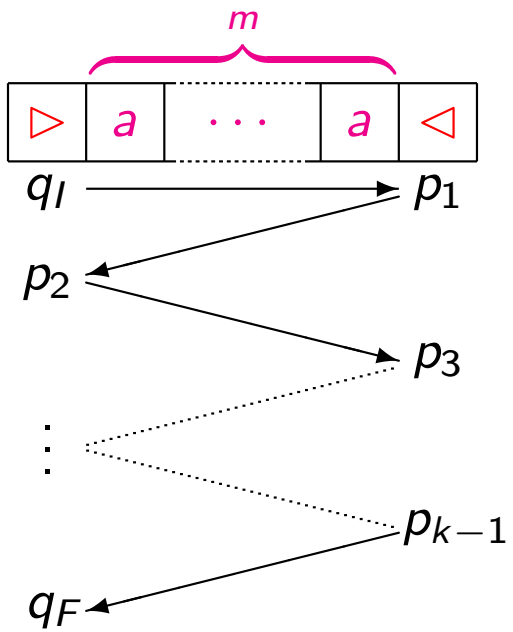
- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$



# A Normal Form for Unary 2NFAs

More details on $M$:

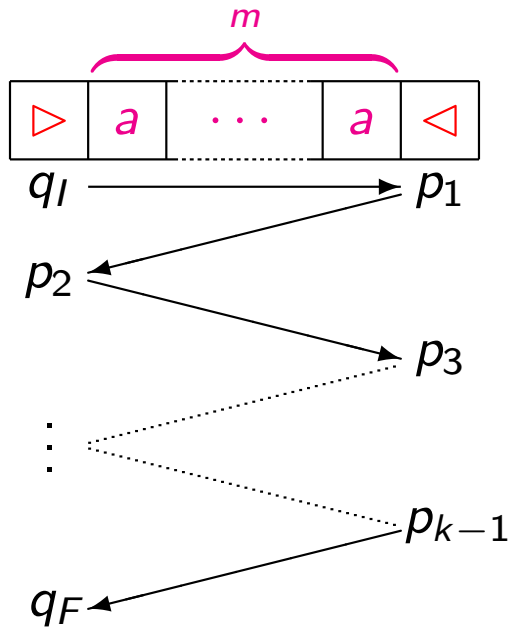
- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$



# A Normal Form for Unary 2NFAs

More details on $M$:

- State set: $\{q_I, q_F\} \cup Q_1 \cup \cdots \cup Q_s$
  - $q_I$ initial state
  - $q_F$ accepting state
  - $Q_i$ deterministic loop of length $\ell_i$
- A computation is a sequence of traversals of the input
- In each traversal $M$ counts the input length modulo one $\ell_i$




**Remark**

If a string is accepted by $M$ then it is accepted by a computation which visits the left end-marker at most $\#Q$ times

# Converting Unary 2NFAs into 2DFAs

[Geffert&Mereghetti&P.'03]

$M$ unary $N$-state 2NFA *in normal form*

$a^m$ input string

- For $p, q \in Q$, $k \geq 1$, we consider the predicate
  $reachable(p, q, k) \equiv$
  $\exists$computation path on $a^m$ which
  - starts in the state $p$ on $\triangleright$
  - ends in the state $q$ on $\triangleright$
  - visits $\triangleright$ at most $k$ times

  Then:
  $a^m \in L(M)$ iff $reachable(q_I, q_F, N)$ is true

- $reachable(p, q, k)$ can be computed by a recursive procedure
- Implemented by a 2DFA with $e^{O(\ln^2 N)}$ states

# Converting Unary 2NFAs into 2DFAs

[Geffert&Mereghetti&P.'03]

$M$ unary $N$-state 2NFA *in normal form*

$a^m$ input string

- For $p, q \in Q$, $k \geq 1$, we consider the predicate $reachable(p, q, k) \equiv$

  $\exists$computation path on $a^m$ which

  - starts in the state $p$ on $\triangleright$
  - ends in the state $q$ on $\triangleright$
  - visits $\triangleright$ at most $k$ times

  Then:
  $a^m \in L(M)$ iff $reachable(q_I, q_F, N)$ is true

- $reachable(p, q, k)$ can be computed by a recursive procedure
- Implemented by a 2DFA with $e^{O(\ln^2 N)}$ states

# Converting Unary 2NFAs into 2DFAs

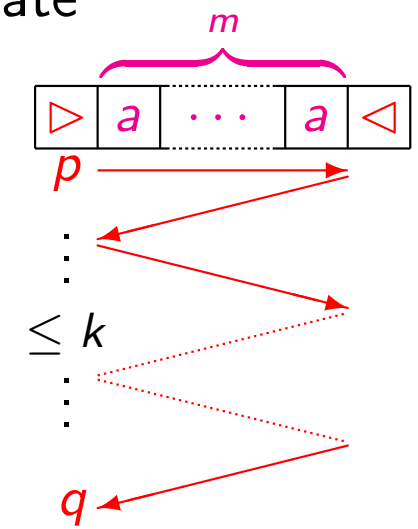
[Geffert&Mereghetti&P.'03]

$M$ unary $N$-state 2NFA *in normal form*

$a^m$ input string

- For $p, q \in Q$, $k \geq 1$, we consider the predicate
  $reachable(p, q, k) \equiv$
  $\exists$computation path on $a^m$ which
  - starts in the state $p$ on $\triangleright$
  - ends in the state $q$ on $\triangleright$
  - visits $\triangleright$ at most $k$ times

  Then:
  $a^m \in L(M)$ iff $reachable(q_I, q_F, N)$ is true




- $reachable(p, q, k)$ can be computed by a recursive procedure
- Implemented by a 2DFA with $e^{O(\ln^2 N)}$ states

# Converting Unary 2NFAs into 2DFAs

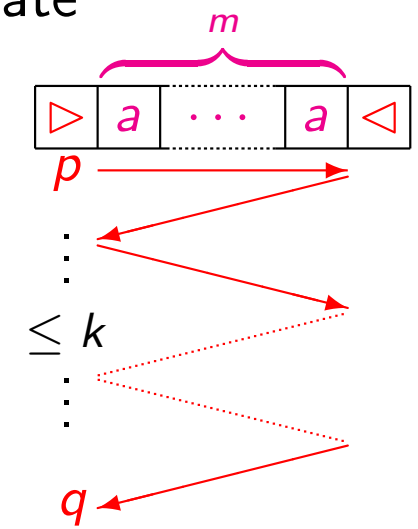
[Geffert&Mereghetti&P.'03]

$M$ unary $N$-state 2NFA *in normal form*

$a^m$ input string

- For $p, q \in Q$, $k \geq 1$, we consider the predicate
  $reachable(p, q, k) \equiv$
  $\exists$computation path on $a^m$ which
  - starts in the state $p$ on $\triangleright$
  - ends in the state $q$ on $\triangleright$
  - visits $\triangleright$ at most $k$ times

  Then:
  $a^m \in L(M)$ iff $reachable(q_I, q_F, N)$ is true




- $reachable(p, q, k)$ can be computed by a recursive procedure
- Implemented by a 2DFA with $e^{O(\ln^2 N)}$ states

# Converting Unary 2NFAs into 2DFAs

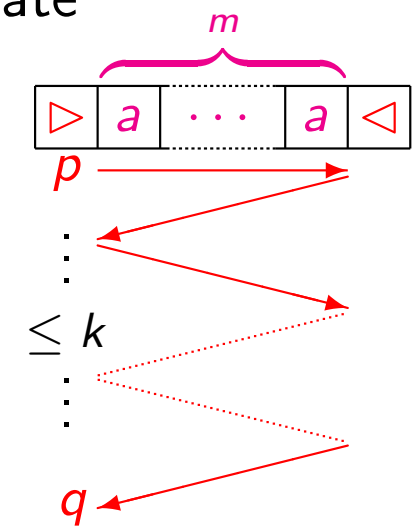
[Geffert&Mereghetti&P.'03]

$M$ unary $N$-state 2NFA *in normal form*

$a^m$ input string

- For $p, q \in Q$, $k \geq 1$, we consider the predicate
  $reachable(p, q, k) \equiv$
  $\exists$computation path on $a^m$ which
  - starts in the state $p$ on $\triangleright$
  - ends in the state $q$ on $\triangleright$
  - visits $\triangleright$ at most $k$ times

  Then:
  $a^m \in L(M)$ iff $reachable(q_I, q_F, N)$ is true




- $reachable(p, q, k)$ can be computed by a recursive procedure
- Implemented by a 2DFA with $e^{O(\ln^2 N)}$ states

# Converting Unary 2NFAs into 2DFAs

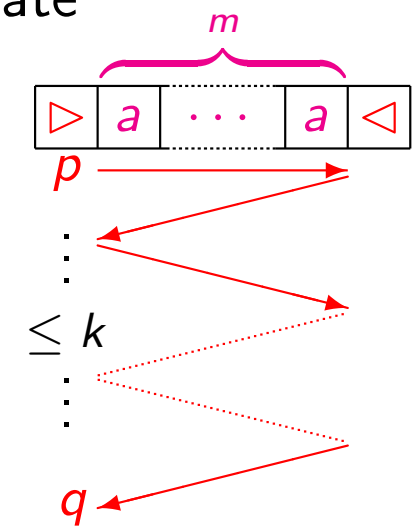
[Geffert&Mereghetti&P.'03]

$M$ unary $N$-state 2NFA *in normal form*

$a^m$ input string

- For $p, q \in Q$, $k \geq 1$, we consider the predicate
  $reachable(p, q, k) \equiv$
  $\exists$computation path on $a^m$ which
  - starts in the state $p$ on $\triangleright$
  - ends in the state $q$ on $\triangleright$
  - visits $\triangleright$ at most $k$ times

  Then:
  $a^m \in L(M)$ iff $reachable(q_I, q_F, N)$ is true




- $reachable(p, q, k)$ can be computed by a recursive procedure
- Implemented by a 2DFA with $e^{O(\ln^2 N)}$ states

# From Unary 2NFAs to 2DFAs

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | |
| $M$ | almost equivalent 2NFA | $N \leq 2n + 2$ states |
| $\Downarrow$ | | |
| $B$ | 2DFA equivalent to $M$ | $e^{O(\ln^2 N)}$ states |
| $\Downarrow$ | | |
| $C$ | 2DFA equivalent to $A$ | $e^{O(\ln^2 n)}$ states |

# From Unary 2NFAs to 2DFAs

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n + 2$ states |
| $\Downarrow$ | | |
| $B$ | 2DFA equivalent to $M$ | $e^{O(\ln^2 N)}$ states |
| $\Downarrow$ | | |
| $C$ | 2DFA equivalent to $A$ | $e^{O(\ln^2 n)}$ states |

# From Unary 2NFAs to 2DFAs

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n + 2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $e^{O(\ln^2 N)}$ states |
| $\Downarrow$ | | |
| $C$ | 2DFA equivalent to $A$ | $e^{O(\ln^2 n)}$ states |

# From Unary 2NFAs to 2DFAs

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n + 2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $e^{O(\ln^2 N)}$ states |
| $\Downarrow$ | Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of $B$ for longer inputs | |
| $C$ | 2DFA equivalent to $A$ | $e^{O(\ln^2 n)}$ states |

# From Unary 2NFAs to 2DFAs

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n+2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $e^{O(\ln^2 N)}$ states |
| $\Downarrow$ | Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of $B$ for longer inputs | |
| $C$ | 2DFA equivalent to $A$ | $e^{O(\ln^2 n)}$ states |

**Theorem ([Geffert&Mereghetti&P.'03])**

*Each unary n-state 2NFA can be simulated by a 2DFA with $e^{O(\ln^2 n)}$ many states*

# From Unary 2NFAs to 2DFAs

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n+2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $e^{O(\ln^2 N)}$ states |
| $\Downarrow$ | Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of $B$ for longer inputs | |
| $C$ | 2DFA equivalent to $A$ | $e^{O(\ln^2 n)}$ states |

**Theorem ([Geffert&Mereghetti&P.'03])**

*Each unary n-state 2NFA can be simulated by a 2DFA with $e^{O(\ln^2 n)}$ many states*

**Upper bound**

- superpolynomial
- subexponential

# From Unary 2NFAs to 2DFAs

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n+2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $e^{O(\ln^2 N)}$ states |
| $\Downarrow$ | Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of $B$ for longer inputs | |
| $C$ | 2DFA equivalent to $A$ | $e^{O(\ln^2 n)}$ states |

**Theorem ([Geffert&Mereghetti&P.'03])**

*Each unary n-state 2NFA can be simulated by a 2DFA with $e^{O(\ln^2 n)}$ many states*

*Can this upper bound be reduced to a polynomial?*

**Upper bound**

- superpolynomial
- subexponential

# Logspace Classes and Graph Accessibility Problem

L: class of languages accepted in logarithmic space by *deterministic* machines

NL: class of languages accepted in logarithmic space by *nondeterministic* machines

Problem

$L \stackrel{?}{=} NL$

# Logspace Classes and Graph Accessibility Problem

L: class of languages accepted in logarithmic space by *deterministic* machines

NL: class of languages accepted in logarithmic space by *nondeterministic* machines

Problem

$L \stackrel{?}{=} NL$

# Logspace Classes and Graph Accessibility Problem

L: class of languages accepted in logarithmic space by *deterministic* machines

NL: class of languages accepted in logarithmic space by *nondeterministic* machines

**Problem**

$L \stackrel{?}{=} NL$

# Logspace Classes and Graph Accessibility Problem

L: class of languages accepted in logarithmic space by *deterministic* machines

NL: class of languages accepted in logarithmic space by *nondeterministic* machines

**Problem**

$L \stackrel{?}{=} NL$

*Graph Accessibility Problem* GAP

- Given $G = (V, E)$ oriented graph, $s, t \in V$
- Decide whether or not $G$ contains a path from $s$ to $t$

# Logspace Classes and Graph Accessibility Problem

L: class of languages accepted in logarithmic space by *deterministic* machines

NL: class of languages accepted in logarithmic space by *nondeterministic* machines

**Problem**

$L \stackrel{?}{=} NL$

*Graph Accessibility Problem* GAP

- Given $G = (V, E)$ oriented graph, $s, t \in V$
- Decide whether or not $G$ contains a path from $s$ to $t$

**Theorem ([Jones '75])**

GAP *is complete for* NL

Hence GAP $\in$ L iff L = NL

# Reduction to GAP

[Geffert&P.'11]

M unary 2NFA in normal form, with $N$ states

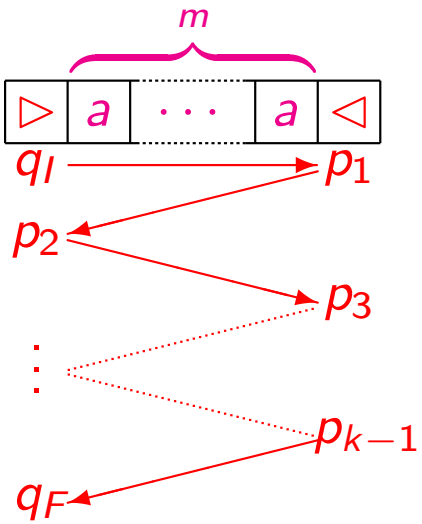
- Accepting computation on $a^m$
  - sequence of traversals of the input
    - starting in $q_I$ on $\triangleright$
    - ending in $q_F$ on $\triangleright$
- Graph $G(m)$
  - vertices $\equiv$ states
  - edges $\equiv$ traversals on $a^m$
- $a^m$ is accepted iff $G(m)$ contains a path from $q_I$ to $q_F$

# Reduction to GAP

[Geffert&P.'11]

M unary 2NFA in normal form, with $N$ states

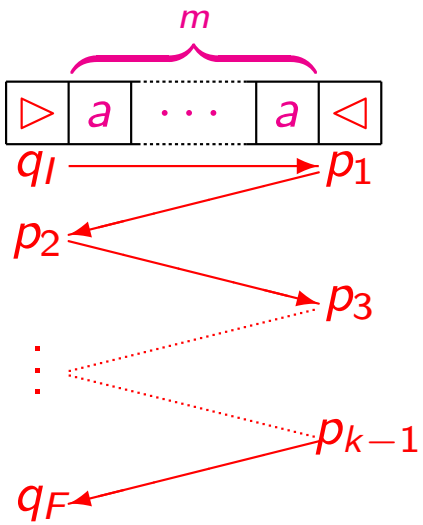
- Accepting computation on $a^m$
  - sequence of traversals of the input
    - starting in $q_I$ on $\triangleright$
    - ending in $q_F$ on $\triangleright$
- Graph $G(m)$
  - vertices $\equiv$ states
  - edges $\equiv$ traversals on $a^m$
- $a^m$ is accepted iff $G(m)$ contains a path from $q_I$ to $q_F$



# Reduction to GAP

[Geffert&P.'11]

M unary 2NFA in normal form, with $N$ states

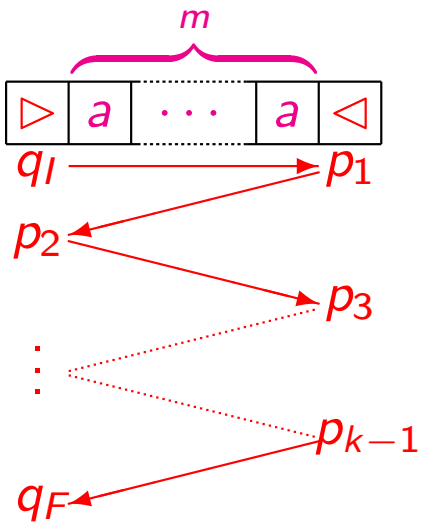



- Accepting computation on $a^m$
  - sequence of traversals of the input
    - starting in $q_I$ on $\triangleright$
    - ending in $q_F$ on $\triangleright$
- Graph $G(m)$
  - vertices $\equiv$ states
  - edges $\equiv$ traversals on $a^m$
- $a^m$ is accepted iff $G(m)$ contains a path from $q_I$ to $q_F$

# Reduction to GAP

[Geffert&P.'11]

M unary 2NFA in normal form, with $N$ states



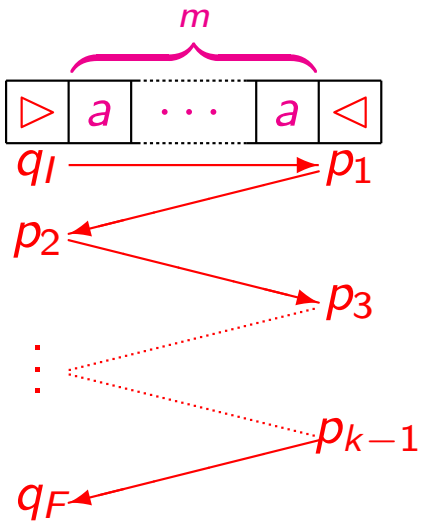

- Accepting computation on $a^m$
  - sequence of traversals of the input
    - starting in $q_I$ on $\triangleright$
    - ending in $q_F$ on $\triangleright$
- Graph $G(m)$
  - vertices $\equiv$ states
  - edges $\equiv$ traversals on $a^m$
- $a^m$ is accepted iff $G(m)$ contains a path from $q_I$ to $q_F$

# Reduction to GAP

[Geffert&P.'11]

M unary 2NFA in normal form, with $N$ states

- Accepting computation on $a^m$
  - sequence of traversals of the input
    - starting in $q_I$ on $\triangleright$
    - ending in $q_F$ on $\triangleright$
- Graph $G(m)$
  - vertices $\equiv$ states
  - edges $\equiv$ traversals on $a^m$

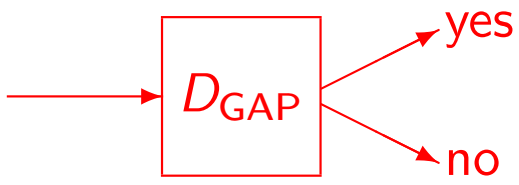



- $a^m$ is accepted iff $G(m)$ contains a path from $q_I$ to $q_F$

*To decide whether or not $a^m \in L(M)$ reduces to decide* GAP *for $G(m)$*

# L = NL $\Rightarrow$ Polynomial Deterministic Simulation!

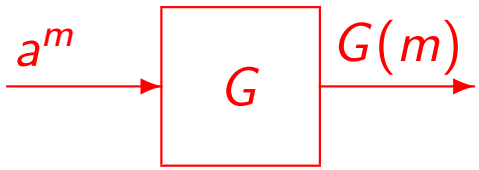
[Geffert&P.'11]




$D_{\mathrm{GAP}}$ logspace bounded *deterministic* machine solving GAP

# L = NL $\Rightarrow$ Polynomial Deterministic Simulation!

[Geffert&P.'11]

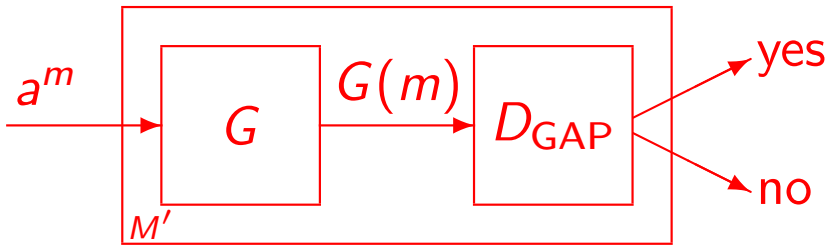



$D_{\mathrm{GAP}}$ logspace bounded *deterministic* machine solving GAP

$G(m)$ graph associated with $a^m$

# L = NL $\Rightarrow$ Polynomial Deterministic Simulation!

[Geffert&P.'11]



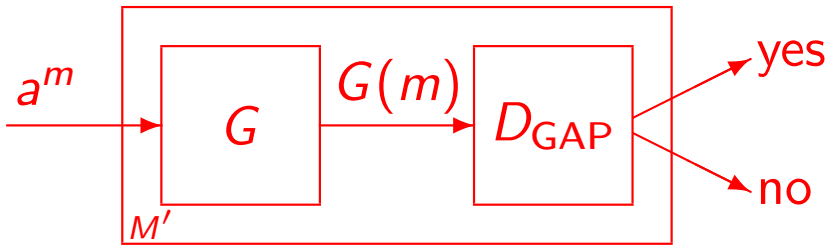

$D_{\mathrm{GAP}}$ logspace bounded *deterministic* machine solving GAP

$G(m)$ graph associated with $a^m$

$M'$ resulting 2DFA

# L = NL $\Rightarrow$ Polynomial Deterministic Simulation!

[Geffert&P.'11]




$D_{\mathrm{GAP}}$ logspace bounded *deterministic* machine solving GAP

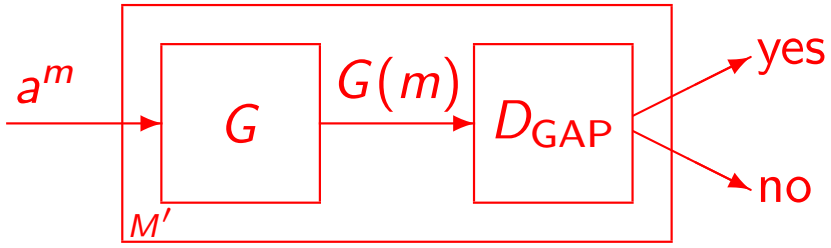
- $O(\log N)$ space — $N$=#states of the given 2NFA $M$
- $poly(N)$ different configurations

$G(m)$ graph associated with $a^m$

$M'$ resulting 2DFA

# L = NL $\Rightarrow$ Polynomial Deterministic Simulation!

[Geffert&P.'11]




$D_{\mathrm{GAP}}$ logspace bounded *deterministic* machine solving GAP

- $O(\log N)$ space — $N$=#states of the given 2NFA $M$
- $poly(N)$ different configurations

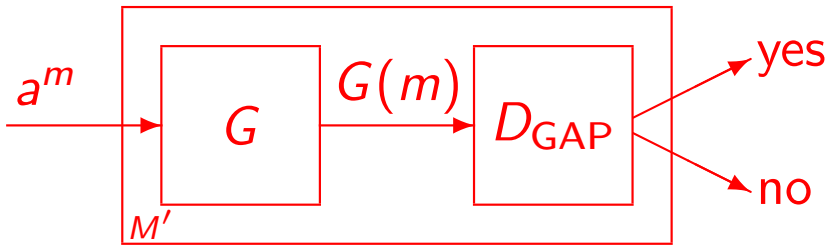
$G(m)$ graph associated with $a^m$

- $O(N^2)$ bits
- $exp(N)$ different configurations — Too many!!!

$M'$ resulting 2DFA

# L = NL $\Rightarrow$ Polynomial Deterministic Simulation!

[Geffert&P.'11]




$D_{\mathrm{GAP}}$ logspace bounded *deterministic* machine solving GAP

- $O(\log N)$ space — $N$=#states of the given 2NFA $M$
- $poly(N)$ different configurations

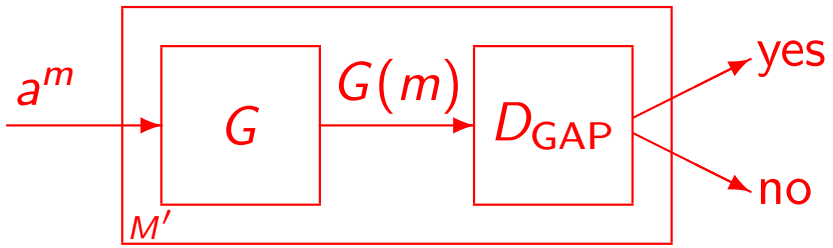
$G(m)$ graph associated with $a^m$

- $O(N^2)$ bits
- $exp(N)$ different configurations — Too many!!!
- bits computed on demand:
  an $N$-state 1DFA $A_{p,q}$ tests the existence of the edge $(p, q)$
  trying to simulate a traversal of $M$ from $p$ to $q$

$M'$ resulting 2DFA

# L = NL $\Rightarrow$ Polynomial Deterministic Simulation!

[Geffert&P.'11]




$D_{\mathrm{GAP}}$ logspace bounded *deterministic* machine solving GAP

- $O(\log N)$ space — $N$=#states of the given 2NFA $M$
- $poly(N)$ different configurations

$G(m)$ graph associated with $a^m$

- $O(N^2)$ bits
- $exp(N)$ different configurations — Too many!!!
- bits computed on demand:
  an $N$-state 1DFA $A_{p,q}$ tests the existence of the edge $(p, q)$
  trying to simulate a traversal of $M$ from $p$ to $q$

$M'$ resulting 2DFA — $poly(N)$ many states!!!

# From Unary 2NFAs to 2DFAs (under L = NL)

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | |
| $M$ | almost equivalent 2NFA | $N \leq 2n + 2$ states |
| $\Downarrow$ | | |
| $B$ | 2DFA equivalent to $M$ | $poly(N)$ states |
| $\Downarrow$ | | |
| $C$ | 2DFA equivalent to $A$ | $poly(n)$ states |

# From Unary 2NFAs to 2DFAs (under L = NL)

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n + 2$ states |
| $\Downarrow$ | | |
| $B$ | 2DFA equivalent to $M$ | $poly(N)$ states |
| $\Downarrow$ | | |
| $C$ | 2DFA equivalent to $A$ | $poly(n)$ states |

# From Unary 2NFAs to 2DFAs (under L = NL)

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n + 2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $poly(N)$ states |
| $\Downarrow$ | | |
| $C$ | 2DFA equivalent to $A$ | $poly(n)$ states |

# From Unary 2NFAs to 2DFAs (under L = NL)

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n + 2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $poly(N)$ states |
| $\Downarrow$ | Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of $B$ for longer inputs | |
| $C$ | 2DFA equivalent to $A$ | $poly(n)$ states |

# From Unary 2NFAs to 2DFAs (under L = NL)

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n+2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $poly(N)$ states |
| $\Downarrow$ | Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of $B$ for longer inputs | |
| $C$ | 2DFA equivalent to $A$ | $poly(n)$ states |

**Theorem ([Geffert&P.'11])**

*If* L = NL *then each unary n-state 2NFA can be simulated by a 2DFA with poly(n) many states*

# From Unary 2NFAs to 2DFAs (under L = NL)

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n+2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $poly(N)$ states |
| $\Downarrow$ | Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of $B$ for longer inputs | |
| $C$ | 2DFA equivalent to $A$ | $poly(n)$ states |

**Theorem ([Geffert&P.'11])**

*If* L = NL *then each unary n-state 2NFA can be simulated by a 2DFA with* $poly(n)$ *many states*

*Proving the conjecture of Sakoda and Sipser for 2NFA → 2DFA in the unary case would separate* L *and* NL *in the general case*

# From Unary 2NFAs to 2DFAs (under L = NL)

| | | |
|---|---|---|
| $A$ | given unary 2NFA | $n$ states |
| $\Downarrow$ | | Conversion into normal form |
| $M$ | almost equivalent 2NFA | $N \leq 2n+2$ states |
| $\Downarrow$ | | Deterministic simulation |
| $B$ | 2DFA equivalent to $M$ | $poly(N)$ states |
| $\Downarrow$ | Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of $B$ for longer inputs | |
| $C$ | 2DFA equivalent to $A$ | $poly(n)$ states |

**Theorem ([Geffert&P.'11])**

*If* L = NL *then each unary n-state 2NFA can be simulated by a 2DFA with* $poly(n)$ *many states*

**Theorem ([Kapoutsis&P.'12])**

L/*poly* $\supseteq$ NL *iff each unary n-state 2NFA can be simulated by a 2DFA with* $poly(n)$ *many states*

# Normal Form for Unary 2NFAs: Consequences

(i) Subexponential simulation of unary 2NFAs by 2DFAs
[Geffert&Mereghetti&P.'03]

(ii) Polynomial simulation of unary 2NFAs by 2DFAs
*under the condition* L = NL [Geffert&P.'11]

(iii) Polynomial simulation of unary 2NFAs by unambiguous 2NFAs
*(unconditional)* [Geffert&P.'11]

(iv) Polynomial complementation of unary 2NFAs
Inductive counting argument [Geffert&Mereghetti&P.'07]

# Normal Form for Unary 2NFAs: Consequences

(i) Subexponential simulation of unary 2NFAs by 2DFAs
[Geffert&Mereghetti&P.'03]

(ii) Polynomial simulation of unary 2NFAs by 2DFAs
*under the condition* $\mathsf{L} = \mathsf{NL}$ [Geffert&P.'11]

(iii) Polynomial simulation of unary 2NFAs by unambiguous 2NFAs
*(unconditional)* [Geffert&P.'11]

(iv) Polynomial complementation of unary 2NFAs
Inductive counting argument [Geffert&Mereghetti&P.'07]

# Normal Form for Unary 2NFAs: Consequences

(i) Subexponential simulation of unary 2NFAs by 2DFAs [Geffert&Mereghetti&P.'03]

(ii) Polynomial simulation of unary 2NFAs by 2DFAs *under the condition* L = NL [Geffert&P.'11]

(iii) Polynomial simulation of unary 2NFAs by unambiguous 2NFAs *(unconditional)* [Geffert&P.'11]

(iv) Polynomial complementation of unary 2NFAs Inductive counting argument [Geffert&Mereghetti&P.'07]

# Normal Form for Unary 2NFAs: Consequences

(i) Subexponential simulation of unary 2NFAs by 2DFAs
[Geffert&Mereghetti&P.'03]

(ii) Polynomial simulation of unary 2NFAs by 2DFAs
*under the condition* L = NL [Geffert&P.'11]

(iii) Polynomial simulation of unary 2NFAs by unambiguous 2NFAs
*(unconditional)* [Geffert&P.'11]

(iv) Polynomial complementation of unary 2NFAs
Inductive counting argument [Geffert&Mereghetti&P.'07]

# Normal Form for Unary 2NFAs: Consequences

(i) Subexponential simulation of unary 2NFAs by 2DFAs [Geffert&Mereghetti&P.'03]

(ii) Polynomial simulation of unary 2NFAs by 2DFAs *under the condition* L = NL [Geffert&P.'11]

(iii) Polynomial simulation of unary 2NFAs by unambiguous 2NFAs *(unconditional)* [Geffert&P.'11]

(iv) Polynomial complementation of unary 2NFAs [Geffert&Mereghetti&P.'07]
Inductive counting argument

Extension to *outer nondeterministic automata*: [Geffert&Guillon&P.'14]

- general alphabet
- unrestricted head reversals
- nondeterministic choices *only* at the endmarkers

# Pushdown Automata and Other Devices

# Unary Context-Free Languages

**Theorem [Ginsurg&Rice '62]**

Each unary context-free languages is regular

*How large could be a finite automaton equivalent to a given unary context-free grammar or pushdown automaton?*

# Unary Context-Free Languages

**Theorem [Ginsurg&Rice '62]**

Each unary context-free languages is regular

*How large could be a finite automaton equivalent to a given unary context-free grammar or pushdown automaton?*

# Unary Pushdown Automata

From PDAs of size $s$, accepting regular languages,
to equivalent 1DFAs

| | unary input | general input |
|---|---|---|
| PDAs | $2^{poly(s)}$ [P.&Shallit&Wang '02] | |
| | | |

All the bounds are tight!

# Unary Pushdown Automata

From PDAs of size $s$, accepting regular languages,
to equivalent 1DFAs

| | unary input | general input |
|---|---|---|
| PDAs | $2^{poly(s)}$ [P.&Shallit&Wang '02] | non recursive [Meyer&Fischer '71] |
| | | |

All the bounds are tight!

# Unary Pushdown Automata

From PDAs of size $s$, accepting regular languages,
to equivalent 1DFAs

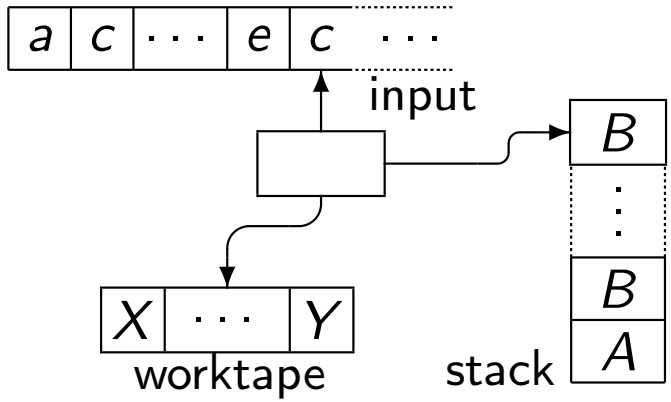
| | unary input | general input |
|---|---|---|
| PDAs | $2^{poly(s)}$ [P.&Shallit&Wang '02] | non recursive [Meyer&Fischer '71] |
| deterministic PDAs | $2^{O(s)}$ [P.'09] | $2^{2^{O(s)}}$ [Valiant '75] |

All the bounds are tight!

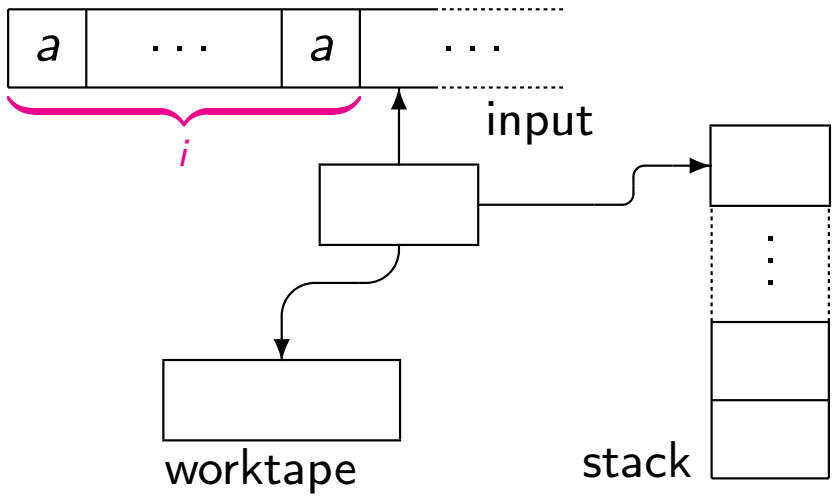
# Auxiliary Pushdown Automata (AuxPDAs)

PDAs augmented with an auxiliary worktape

'SPACE' $\equiv$ worktape

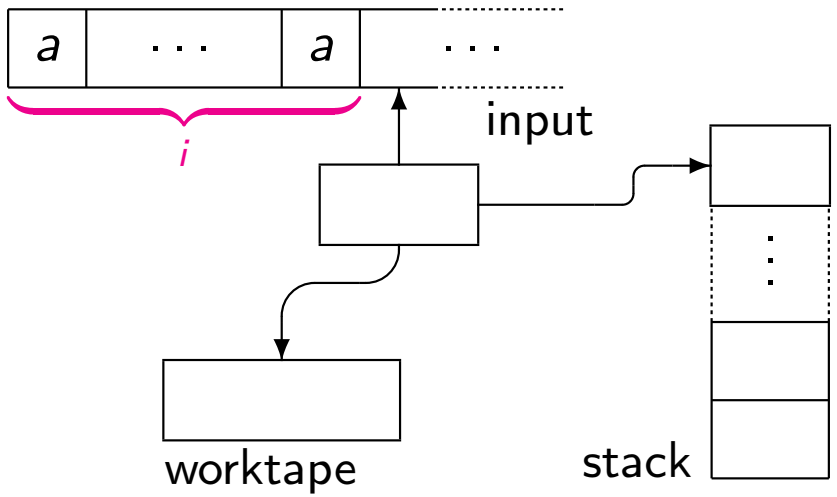


# 1AuxPDAs: How to Count the Input Length



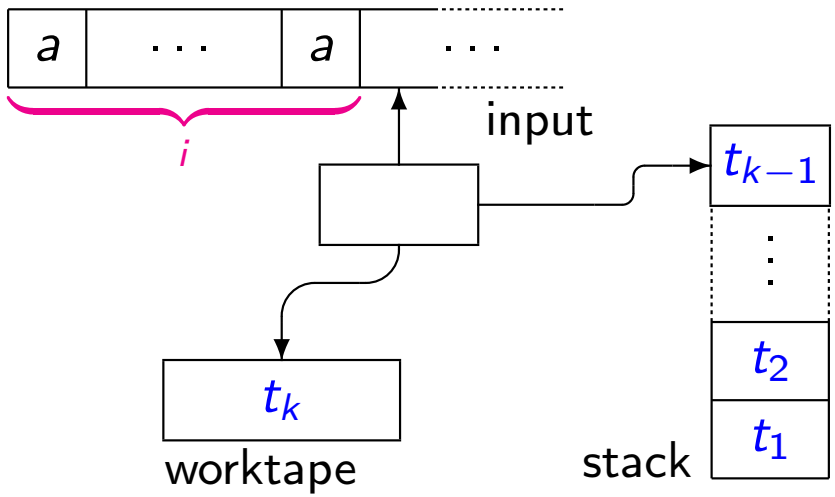

$i$

# 1AuxPDAs: How to Count the Input Length




$i = (1\ 1\ 0\ \cdots\ 1\ 0\ 0\ 0\ 1\ 0)_2$

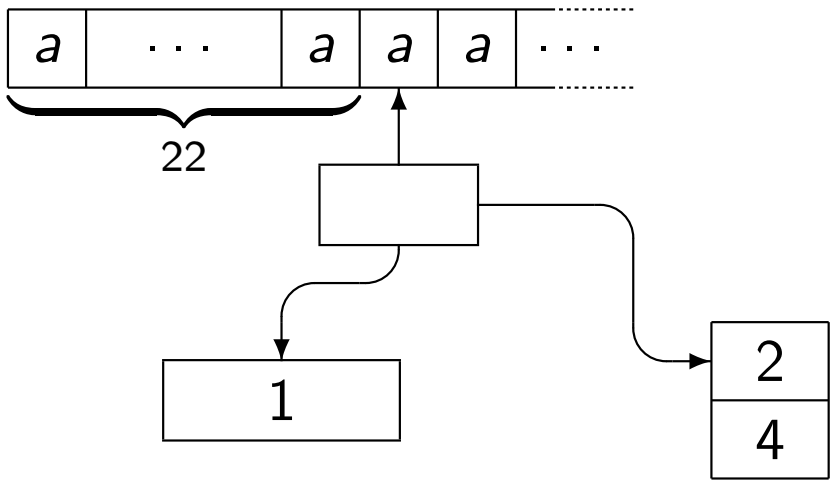
# 1AuxPDAs: How to Count the Input Length




$$i = (\underset{t_1}{1}\ \underset{t_2}{1}\ 0\ \cdots\ \underset{t_{k-1}}{1}\ 0\ 0\ 0\ \underset{t_k}{1}\ 0)_2 = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_{k-1}} + 2^{t_k}$$

# 1AuxPDAs: How to Count the Input Length

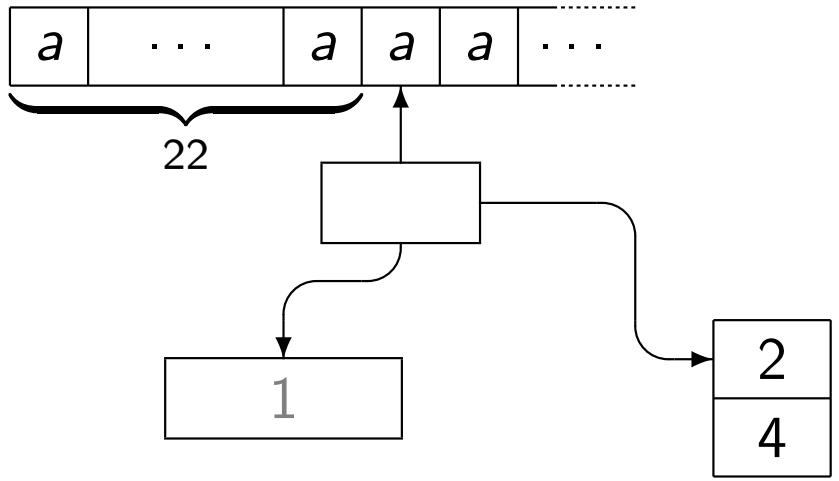
$$22 = 2^4 + 2^2 + 2^1$$



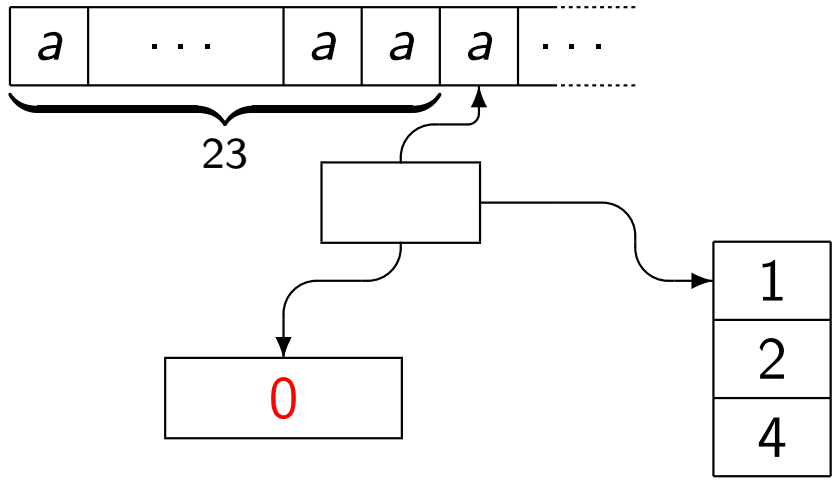

$$23 = 2^4 + 2^2 + 2^1 + 2^0$$

# 1AuxPDAs: How to Count the Input Length

$22 = 2^4 + 2^2 + 2^1$

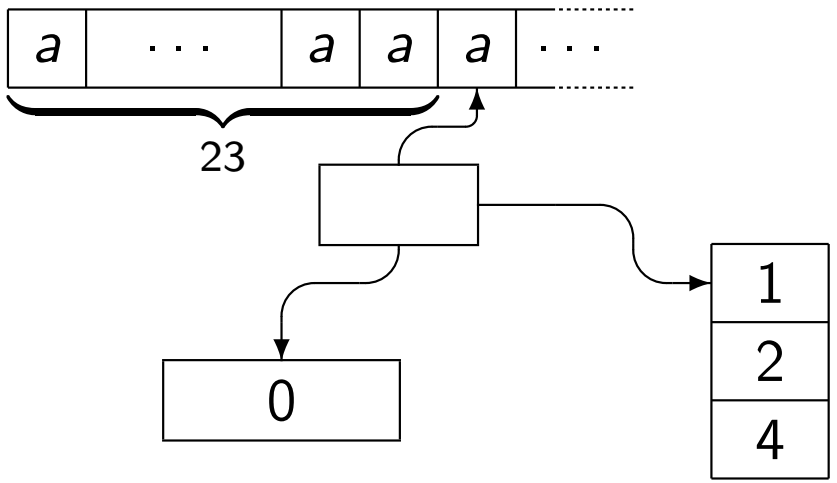



$23 = 2^4 + 2^2 + 2^1 + 2^0$



# 1AuxPDAs: How to Count the Input Length

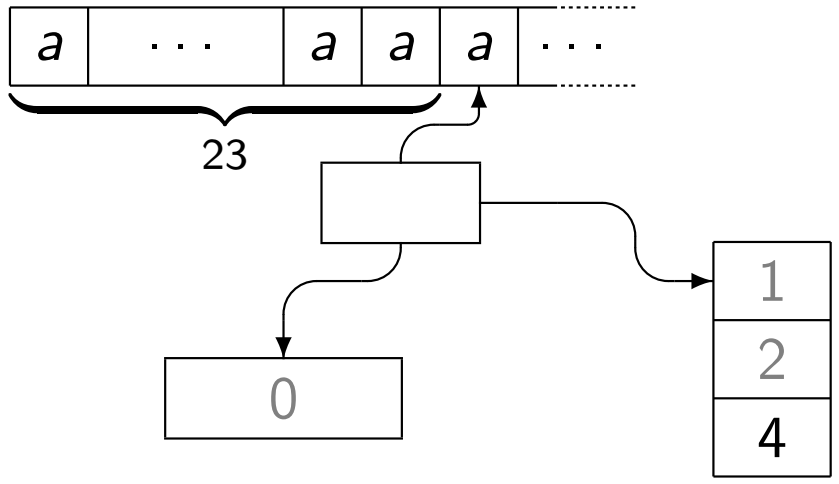
$$23 = 2^4 + 2^2 + 2^1 + 2^0$$



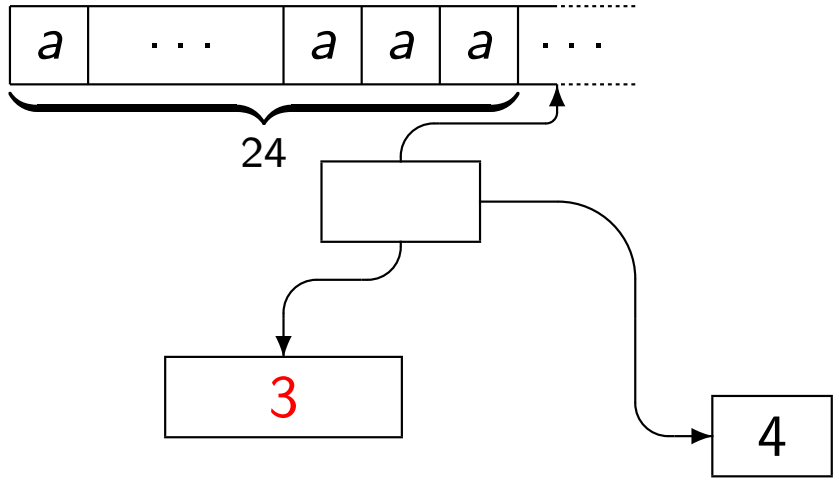

$$24 = 2^4 + 2^3$$

# 1AuxPDAs: How to Count the Input Length

$23 = 2^4 + 2^2 + 2^1 + 2^0$




$24 = 2^4 + 2^3$



# Example: $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$

- $\mathcal{L}_p$ is nonregular
- $\mathcal{L}_p$ is accepted by a 1AuxPDA $M$ which:
  - scans the input while counting its length
  - accepts iff the pushdown store is empty
    i.e., the binary representation of the input length contains exactly one digit 1
- On input $a^n$
  the largest integer stored on the worktape is $\lfloor \log_2 n \rfloor$,
  which is represented in $O(\log \log n)$ space

# Example: $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$

- $\mathcal{L}_p$ is nonregular
- $\mathcal{L}_p$ is accepted by a 1AuxPDA $M$ which:
  - scans the input while counting its length
  - accepts iff the pushdown store is empty
    i.e., the binary representation of the input length contains exactly one digit 1
- On input $a^n$
  the largest integer stored on the worktape is $\lfloor \log_2 n \rfloor$,
  which is represented in $O(\log \log n)$ space

# Example: $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$

- $\mathcal{L}_p$ is nonregular
- $\mathcal{L}_p$ is accepted by a 1AuxPDA $M$ which:
  - scans the input while counting its length
  - accepts iff the pushdown store is empty
    i.e., the binary representation of the input length contains exactly one digit 1
- On input $a^n$
  the largest integer stored on the worktape is $\lfloor \log_2 n \rfloor$,
  which is represented in $O(\log \log n)$ space

# Example: $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$

- $\mathcal{L}_p$ is nonregular
- $\mathcal{L}_p$ is accepted by a 1AuxPDA $M$ which:
  - scans the input while counting its length
  - accepts iff the pushdown store is empty
    i.e., the binary representation of the input length contains exactly one digit 1
- On input $a^n$
  the largest integer stored on the worktape is $\lfloor \log_2 n \rfloor$,
  which is represented in $O(\log \log n)$ space

$$\mathcal{L}_p \in \text{1AuxPDASpace}(\log \log n)$$

# Space Bounds on 1AuxPDAs

$\mathcal{L}_p$ is accepted using the *minimum amount of space* for nonregular languages recognition:

**Theorem ([P.&Shallit&Wang '02])**

*If a unary language $L$ is accepted by a 1AuxPDA in $o(\log \log n)$ space then $L$ is regular*

In contrast

- with a binary alphabet,
- and space measured on the 'less' expensive accepting computation:

**Theorem ([Chytil '86])**

*For each $k \geq 2$ there is a non context-free language $L_k$ accepted by a 1AuxPDA in $O(\underbrace{\log \ldots \log}_{k} n)$ space*

# Space Bounds on 1AuxPDAs

$\mathcal{L}_p$ is accepted using the *minimum amount of space* for nonregular languages recognition:

**Theorem ([P.&Shallit&Wang '02])**

*If a unary language $L$ is accepted by a 1AuxPDA in $o(\log\log n)$ space then $L$ is regular*

In contrast

- with a binary alphabet,
- and space measured on the 'less' expensive accepting computation:

**Theorem ([Chytil '86])**

*For each $k \geq 2$ there is a non context-free language $L_k$ accepted by a 1AuxPDA in $O(\underbrace{\log \ldots \log}_{k} n)$ space*

# Two-way Pushdown Automata (2PDAs)

- More powerful than PDAs, e.g., $\{a^n b^n c^n \mid n \geq 0\}$
- 2DPDAs can be simulated by RAMs in *linear time* [Cook '71]

Main open problems:

- Power of nondeterminism, i.e., 2DPDAs vs 2PDA
- 2DPDAs vs linear bounded automata

# Two-way Pushdown Automata (2PDAs)

- More powerful than PDAs, e.g., $\{a^n b^n c^n \mid n \geq 0\}$
- 2DPDAs can be simulated by RAMs in *linear time* [Cook '71]

Main open problems:

- Power of nondeterminism, i.e., 2DPDAs vs 2PDA
- 2DPDAs vs linear bounded automata

# Unary 2PDAs

- Very powerful models, even in the deterministic version

**Theorem ([Monien '84])**

*The unary encoding of each language in* P *is accepted by a 2DPDA*

- With a *constant number* of input head reversals they accept only regular languages [Liu&Weiner '68]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$ accepted by a 2DPDA making $\approx \log_2 n$ reversals

# Unary 2PDAs

- Very powerful models, even in the deterministic version

**Theorem ([Monien '84])**

*The unary encoding of each language in* P *is accepted by a 2DPDA*

- With a *constant number* of input head reversals they accept only regular languages [Liu&Weiner '68]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$ accepted by a 2DPDA making $\approx \log_2 n$ reversals

# Unary 2PDAs

- Very powerful models, even in the deterministic version

**Theorem ([Monien '84])**

*The unary encoding of each language in* P *is accepted by a 2DPDA*

- With a *constant number* of input head reversals they accept only regular languages [Liu&Weiner '68]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$ accepted by a 2DPDA making $\approx \log_2 n$ reversals

# Unary 2PDAs

- Very powerful models, even in the deterministic version

**Theorem ([Monien '84])**

*The unary encoding of each language in* P *is accepted by a 2DPDA*

- With a *constant number* of input head reversals they accept only regular languages [Liu&Weiner '68]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$ accepted by a 2DPDA making $\approx \log_2 n$ reversals

**Problem**

Does there exist a unary nonregular language accepted by a 2PDA making $o(\log n)$ head reversals?

# Multi-Head Finite Automata

- More powerful than one-head finite automata,
  even if the heads are *one-way*, e.g., $\{a^n b^n \mid n \geq 0\}$
- *Unary case:*
  with a *constant number* of head reversals
  they accept only regular languages [Sudborough '74]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$
  accepted by a 2-head automaton making $\approx \log_2 n$ reversals

# Multi-Head Finite Automata

- More powerful than one-head finite automata,
  even if the heads are *one-way*, e.g., $\{a^n b^n \mid n \geq 0\}$
- *Unary case:*
  with a *constant number* of head reversals
  they accept only regular languages [Sudborough '74]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$
  accepted by a 2-head automaton making $\approx \log_2 n$ reversals

# Multi-Head Finite Automata

- More powerful than one-head finite automata,
  even if the heads are *one-way*, e.g., $\{a^n b^n \mid n \geq 0\}$
- *Unary case:*
  with a *constant number* of head reversals
  they accept only regular languages [Sudborough '74]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$
  accepted by a 2-head automaton making $\approx \log_2 n$ reversals

# Multi-Head Finite Automata

- More powerful than one-head finite automata,
  even if the heads are *one-way*, e.g., $\{a^n b^n \mid n \geq 0\}$
- *Unary case:*
  with a *constant number* of head reversals
  they accept only regular languages [Sudborough '74]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$
  accepted by a 2-head automaton making $\approx \log_2 n$ reversals

**Problem**

Does there exist a unary nonregular language
accepted by a multi-head automaton
making $o(\log n)$ head reversals?

# Multi-Head Finite Automata

- More powerful than one-head finite automata,
  even if the heads are *one-way*, e.g., $\{a^n b^n \mid n \geq 0\}$
- *Unary case:*
  with a *constant number* of head reversals
  they accept only regular languages [Sudborough '74]
- $\mathcal{L}_p = \{a^{2^m} \mid m \geq 0\}$
  accepted by a 2-head automaton making $\approx \log_2 n$ reversals

**Problem**

Does there exist a unary nonregular language
accepted by a multi-head automaton
making $o(\log n)$ head reversals?

- Unary multi-head 2PDAs making $O(1)$ input head reversals
  accept only regular languages [Ibarra '74]

# Conclusion

# Final Remarks

Unary Automata and Languages

- Interesting properties and differences with respect to the general case
- Special methods (e.g., from number theory)
- Important relationships with the general case
- Several open problems

Thank you for your attention!