

Optimal State Reductions of Automata with Partially Specified Behaviors

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Motivation

Finite automaton A

Question: $x \in L(A)$?

Answer: Yes/No

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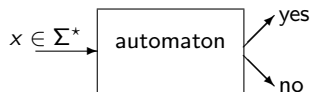
Goal: Study automata with three kinds of states:

- ▶ *accepting* states
- ▶ *rejecting* states
- ▶ *don't care* states

Motivation: Example

$$\Sigma = \{-, 0, \dots, 9\}$$

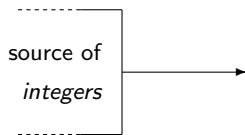
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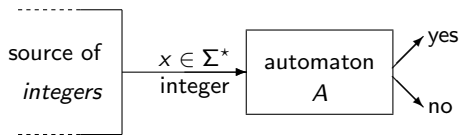
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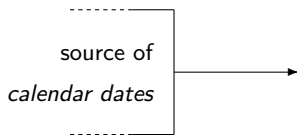


We do not care the behavior of A on strings that do not represent integers!

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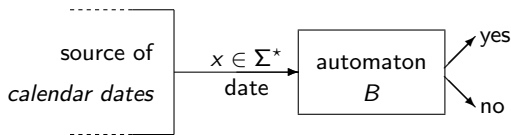
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Some strings: ~~12~~ ~~9--~~ ~~-12~~ 27-01-2015 ...



We do not care the behavior of B on strings that do not represent dates!

Related Works

Digital system design: incomplete Moore machines

- ▶ Minimization (Paull & Unger, 1959)
- ▶ NP-hardness (Pfleeger, 1973)
- ▶ several exact and heuristic algorithms since then

Model checking:

- ▶ Automata with three color states
(Chen. et al, 2009)
- ▶ Automata over infinite words
(Eisinger & Klaedtke, 2008)

Automata theory:

- ▶ Self-verifying automata
(Jirásková & Pighizzini, 2011)

Automata with *don't care* States (dcNFAs)

$$A = \langle Q, \Sigma, \delta, I, F^{\oplus}, F^{\ominus} \rangle$$

- ▶ nondeterministic transitions and multiple initial states
- ▶ F^{\oplus} **accepting states**
- ▶ F^{\ominus} **rejecting states**

Languages:

- ▶ $\mathcal{L}^{\oplus}(A)$ **accepted language**
- ▶ $\mathcal{L}^{\ominus}(A)$ **rejected language**

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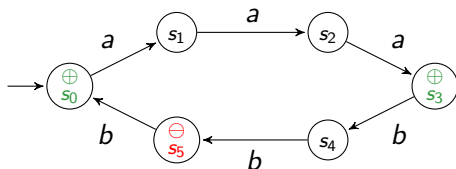
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Special case: *self-verifying automata*

- ▶ when $\mathcal{L}^{\oplus}(A) \cup \mathcal{L}^{\ominus}(A) = \Sigma^*$

Example



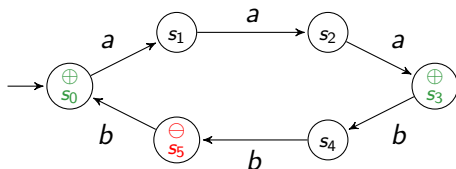
- ▶ $\mathcal{L}^{\oplus}(A) = (a^3 b^3)^*(\epsilon + a^3)$ accepted language
- ▶ $\mathcal{L}^{\ominus}(A) = (a^3 b^3)^*(a^3 b^2)$ rejected language

Definition 1

A language L is said to be **compatible** with a dcNFA A whenever

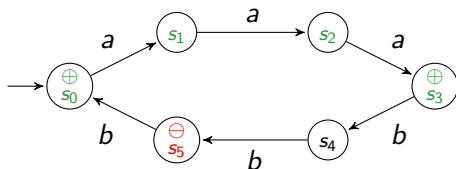
$$\mathcal{L}^{\oplus}(A) \subseteq L \quad \text{and} \quad \mathcal{L}^{\ominus}(A) \subseteq L^c$$

Example



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- ▶ $\mathcal{L}^{\ominus}(A) = (a^3 b^3)^*(a^3 b^2)$ rejected language
- ▶ $L = (a^3 b^3)^*(\epsilon + a + a^2 + a^3)$ is compatible with A

Conversion into Compatible DFAs

Compatibility Graph

$A = \langle Q, \Sigma, \delta, I, F^\oplus, F^\ominus \rangle$ dcNFA

- ▶ L_q^\oplus, L_q^\ominus languages accepted and rejected *starting from* $q \in Q$
- ▶ $p, q \in Q$ are **compatible** iff $(L_p^\oplus \cup L_q^\oplus) \cap (L_p^\ominus \cup L_q^\ominus) = \emptyset$

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Compatibility graph of A:

- ▶ the vertex set is Q
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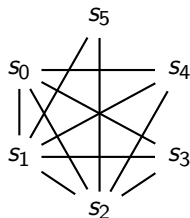
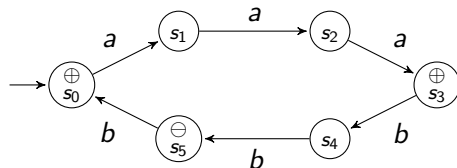
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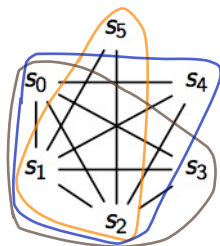
Example:



Clique Covering

A *clique covering* of an undirected graph $G = (Q, E)$ is a set $\{\alpha_1, \dots, \alpha_s\}$ s.t.

- ▶ $\alpha_i \subseteq Q, i = 1, \dots, s$
- ▶ the graph $(\alpha_i, E \cap (\alpha_i \times \alpha_i))$ is *complete*, $i = 1, \dots, s$
- ▶ $\bigcup_{i=1}^s \alpha_i = Q$



$\{s_0, s_1, s_2, s_3\}$

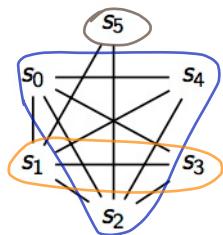
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$$\begin{aligned} &\{s_0, s_2, s_4\} \\ &\{s_1, s_3\} \\ &\{s_5\} \end{aligned}$$

Characterization Theorem

Given:

- ▶ dcNFA $A = \langle Q, \Sigma, \delta, I, F^\oplus, F^\ominus \rangle$
- ▶ DFAs $A' = \langle Q', \Sigma, \delta', i', F' \rangle$

A' is compatible with A iff there is a function $\phi : Q' \rightarrow 2^Q$ s.t.

$\phi(Q')$ is a *clique covering* of the compatibility graph of A

and ... (details in the proceedings)

A Pseudo-Subset Construction

$A = \langle Q, \Sigma, \delta, I, F^{\oplus}, F^{\ominus} \rangle$ a given a dcNFA

Define a DFA $A' = \langle Q', \Sigma, \delta', i', F' \rangle$ as:

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$\delta'(\alpha, \sigma)$ = a clique that includes all the states reachable from states in $\alpha \in Q'$ reading $\sigma \in \Sigma$, i.e.,

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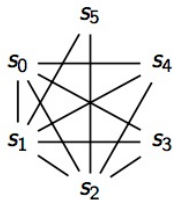
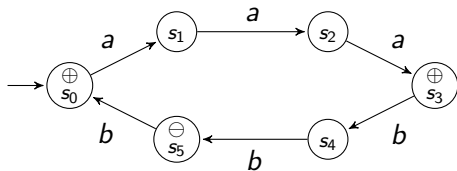
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F' = a set satisfying:

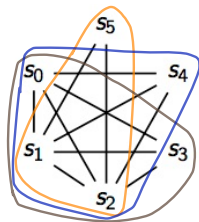
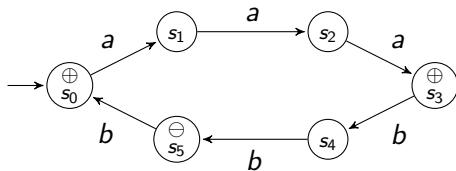
- all cliques containing accepting states are in F' , and
- all cliques containing rejecting states are not in F' ,
i.e.

$$\{\alpha \mid \alpha \cap F^\oplus \neq \emptyset\} \subseteq F' \subseteq \{\alpha \mid \alpha \cap F^\ominus = \emptyset\}$$

dcNFA and Compatible DFAs: Example

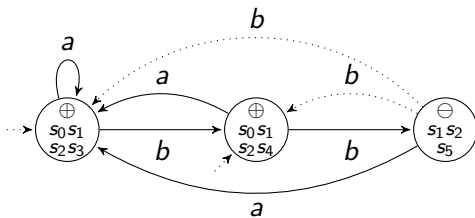


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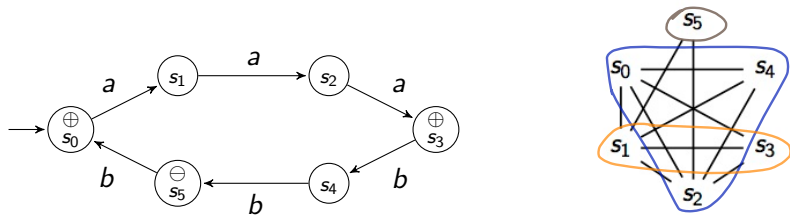


3 maximal cliques

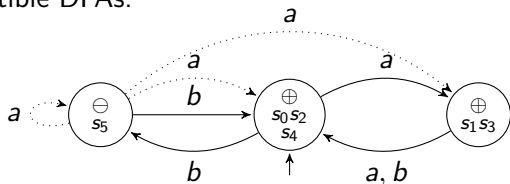
DFAs obtained with the pseudo-subset construction:



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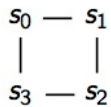
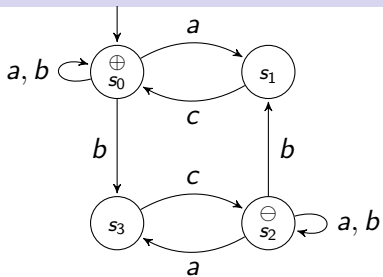


More compatible DFAs:

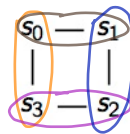
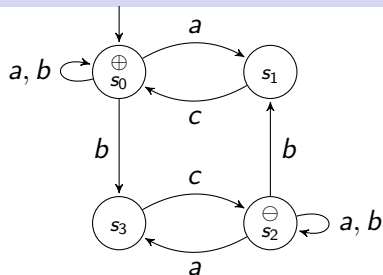


Furthermore, there are no compatible DFAs with < 3 states!

Another Example

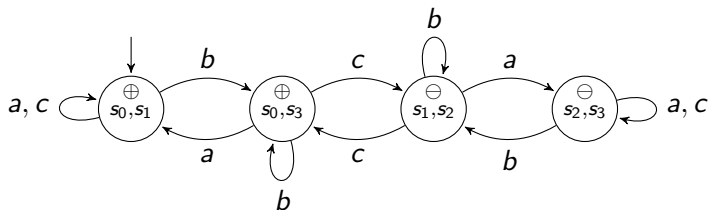


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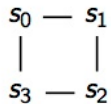
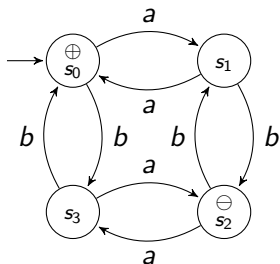
4 maximal cliques

Pseudo-subset construction:

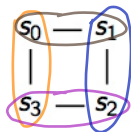
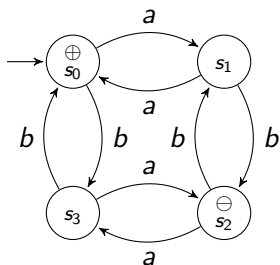


In this example, all compatible DFAs require at least 4 states!

Covering without Maximal Cliques



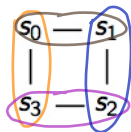
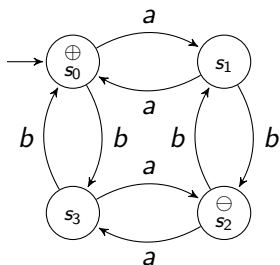
Covering without Maximal Cliques



4 maximal cliques

The pseudo-subset construction produces DFAs with 4 states

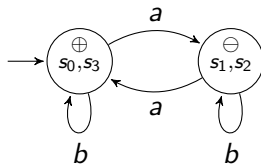
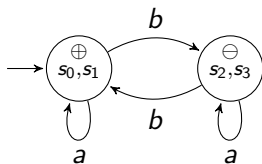
Covering without Maximal Cliques



4 maximal cliques

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However we can do better, using coverings with two cliques!



Size Bounds of Smallest Compatible DFAs

Theorem 2

Let A be a dcNFA:

- ▶ *There exists a compatible DFA whose number of states is bounded by the number of maximal cliques in the compatibility graph of A*

Upper bound:

Number of maximal cliques in the compatibility graph

- ▶ *Each DFA compatible with A should have at least as many states as the smallest number of cliques covering the compatibility graph of A*

Lower bound:

Minimum number of cliques covering the compatibility graph

State Complexity

State Complexity in the General Case

Theorem 3

For each n -state dcNFA ($n \geq 2$)
there exists a compatible DFA with at most $f(n)$ states, s.t.

$$f(n) = \begin{cases} 3^{\lfloor n/3 \rfloor}, & \text{if } n \equiv 0 \pmod{3}, \\ 4 \cdot 3^{\lfloor n/3 \rfloor - 1}, & \text{if } n \equiv 1 \pmod{3}, \\ 2 \cdot 3^{\lfloor n/3 \rfloor}, & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Furthermore this bound can be effectively reached

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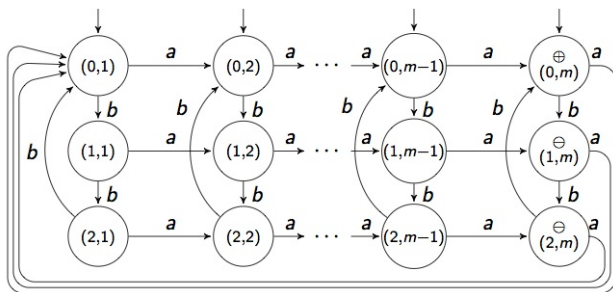
Upper bound:

$f(n)$ is the maximal number of maximal cliques in a graph with n vertices (Moon & Moser, 1965)

Proof

Lower bound:

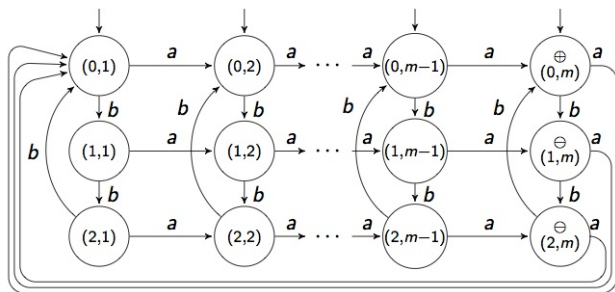
from the lower bound for the conversion of *self-verifying automata* into DFAs (Jirásková & Pighizzini, 2011)



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Lower bound:

from the lower bound for the conversion of *self-verifying automata* into DFAs (Jirásková & Pighizzini, 2011)



With a *single initial state* (but *nondeterministic transitions*), the optimal state bound remains the same

State Complexity in the Deterministic Case

Let A be an n -state dcDFA

- ▶ There exists a compatible DFA with n states which is obtained by arbitrarily marking each don't care state either as accepting or as rejecting

- ▶ This bound cannot be reduced

Worst case:

A does not contain any *don't care* state and it is minimal

Time Complexity

NP-completeness

Theorem 4

The following problem is NP-complete:

*Given a dcNFA A and an integer $k > 0$,
does there exist a compatible DFA with $\leq k$ states?*

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In polynomial time we can

- ▶ nondeterministically generate a DFA B with $\leq k$ states
- ▶ verify that B is compatible with A , as follows:
for each reachable state (p, q) in the “product” of A and B
the following conditions should be verified
 - ▶ if p accepting in A then q is final in B
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NP-hardness follows from (Pfleeger, 1973)
(even if A is a dcDFA!)



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Minimization of dcDFAs and dcNFAs is NP-complete

Conclusion

Our Contributions

- ▶ Characterization of DFAs compatible with each given dcNFA
- ▶ Pseudo-subset construction
- ▶ Upper and lower bounds for the number of states of the smallest compatible DFA
- ▶ NP-completeness of the reduction of dcDFAs and dcNFAs to minimal compatible DFAs
- ▶ dcNFAs over a one-letter alphabet

Some Possible Future Investigations

- ▶ Classes of dcNFAs with compatible DFAs of polynomial size
- ▶ Operations on dcNFAs and their state complexity
- ▶ Extension of *don't care* notion to other devices
- ▶ ...

Thank you for your attention!