Optimal State Reductions of Automata with Partially Specified Behaviors

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Motivation

Finite automaton A

Question: $x \in L(A)$?

Answer: Yes/No

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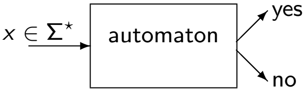
Goal: Study automata with three kinds of states:

- *accepting* states
- *rejecting* states
- *don't care* states

Motivation: Example

$\Sigma = \{\texttt{-}, \texttt{0}, \ldots, \texttt{9}\}$

Some strings: `12` `9--` `-12` `27-01-2015` ...



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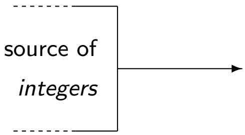

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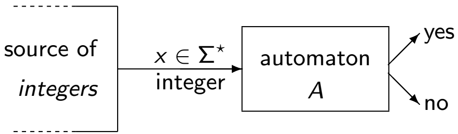

We do not care the behavior of A on strings that do not represent integers!

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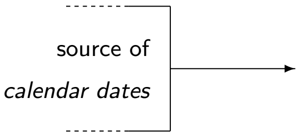

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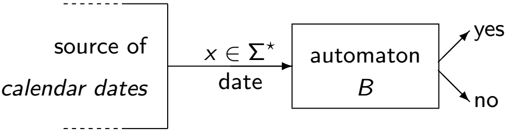

We do not care the behavior of B on strings that do not represent dates!

Related Works

Digital system design: incomplete Moore machines

- Minimization (Paull & Unger, 1959)
- NP-hardness (Pfleeger, 1973)
- several exact and heuristic algorithms since then

Model checking:

- Automata with three color states (Chen. et al, 2009)
- Automata over infinite words (Eisinger & Klaedtke, 2008)

Automata theory:

- Self-verifying automata (Jirásková & Pighizzini, 2011)

Automata with *don't care* States (dcNFAs)

$A = \langle Q, \Sigma, \delta, I, F^{\oplus}, F^{\ominus} \rangle$

- nondeterministic transitions and multiple initial states
- $F^{\oplus}$ accepting states
- $F^{\ominus}$ rejecting states

Languages:

- $\mathcal{L}^{\oplus}(A)$ accepted language
- $\mathcal{L}^{\ominus}(A)$ rejected language

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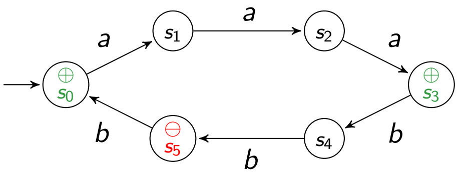
Special case: *self-verifying automata*

- when $\mathcal{L}^{\oplus}(A) \cup \mathcal{L}^{\ominus}(A) = \Sigma^{\star}$

Example



- $\mathcal{L}^{\oplus}(A) = (a^3 b^3)^{\star}(\varepsilon + a^3)$ accepted language
- $\mathcal{L}^{\ominus}(A) = (a^3 b^3)^{\star}(a^3 b^2)$ rejected language

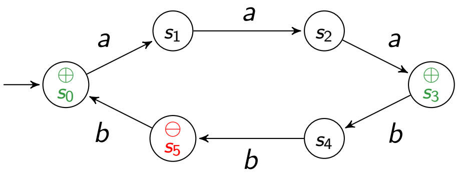
Compatibility

Definition 1

A language L is said to be compatible with a dcNFA A whenever

$$\mathcal{L}^{\oplus}(A) \subseteq L \quad \text{and} \quad \mathcal{L}^{\ominus}(A) \subseteq L^c$$

Example



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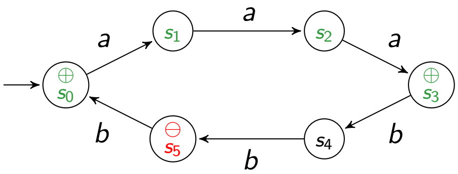

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- $\mathcal{L}^{\ominus}(A) = (a^3b^3)^\star(a^3b^2)$ rejected language
- $L = (a^3b^3)^\star(\varepsilon + a + a^2 + a^3)$ is compatible with A

Conversion into Compatible DFAs

Compatibility Graph

$A = \langle Q, \Sigma, \delta, I, F^{\oplus}, F^{\ominus} \rangle$ dcNFA

- $L_q^{\oplus}$, $L_q^{\ominus}$ languages accepted and rejected *starting from* $q \in Q$
- $p, q \in Q$ are compatible iff $(L_p^{\oplus} \cup L_q^{\oplus}) \cap (L_p^{\ominus} \cup L_q^{\ominus}) = \emptyset$

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- the vertex set is Q
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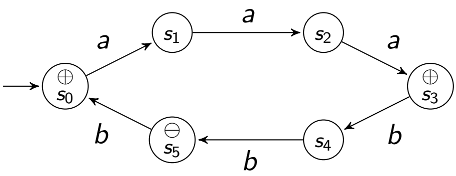
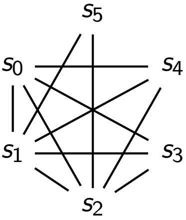
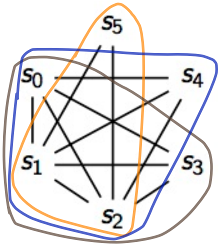




Clique Covering

A *clique covering* of an undirected graph $G = (Q, E)$ is a set $\{\alpha_1, \dots, \alpha_s\}$ s.t.

- $\alpha_i \subseteq Q$, $i = 1, \dots, s$
- the graph $(\alpha_i, E \cap (\alpha_i \times \alpha_i))$ is *complete*, $i = 1, \dots, s$
- $\bigcup_{i=1}^{s} \alpha_i = Q$



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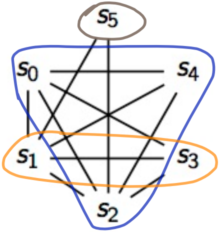

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Characterization Theorem

Given:

- dcNFA $A = \langle Q, \Sigma, \delta, I, F^{\oplus}, F^{\ominus} \rangle$
- DFAs $A' = \langle Q', \Sigma, \delta', i', F' \rangle$

A' is compatible with A iff there is a function $\phi : Q' \rightarrow 2^{Q}$ s.t.

$\phi(Q')$ is a *clique covering* of the compatibility graph of A

and ... (details in the proceedings)

A Pseudo-Subset Construction

$A = \langle Q, \Sigma, \delta, I, F^{\oplus}, F^{\ominus} \rangle$ a given a dcNFA

Define a DFA $A' = \langle Q', \Sigma, \delta', i', F' \rangle$ as:

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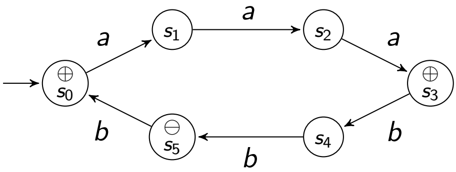
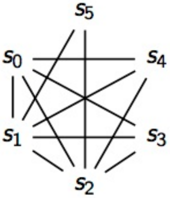
$F' =$ a set satisfying:

- all cliques containing accepting states are in F', and
- all cliques containing rejecting states are not in F', i.e.

$$\{\alpha \mid \alpha \cap F^{\oplus} \neq \emptyset\} \subseteq F' \subseteq \{\alpha \cap F^{\ominus} = \emptyset\}$$

dcNFA and Compatible DFAs: Example





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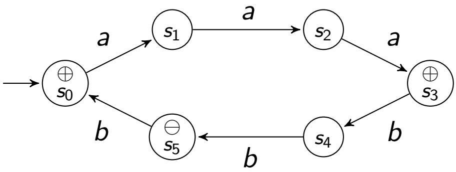
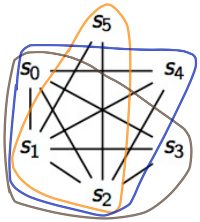
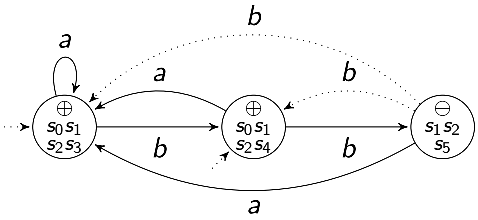




3 maximal cliques

DFAs obtained with the pseudo-subset construction:



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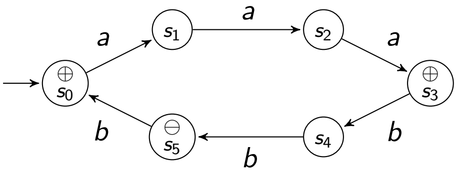
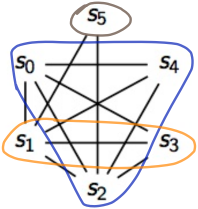
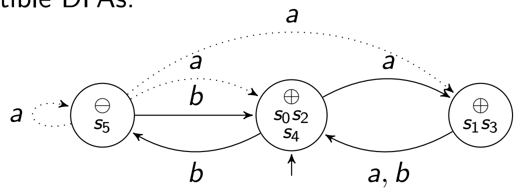


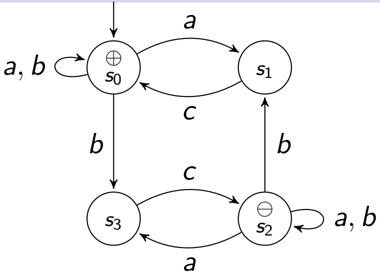


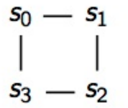
More compatible DFAs:



Furthermore, there are no compatible DFAs with < 3 states!

Another Example





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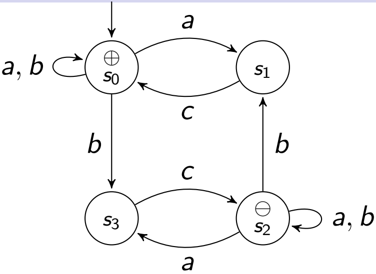
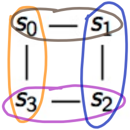
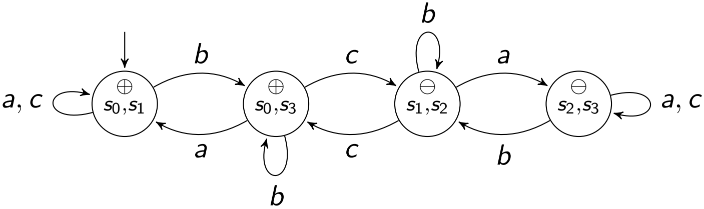




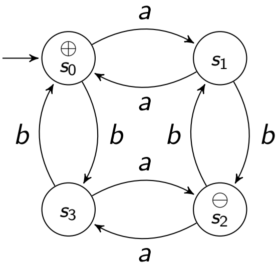
4 maximal cliques

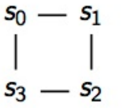
Pseudo-subset construction:



In this example, all compatible DFAs require at least 4 states!

Covering without Maximal Cliques





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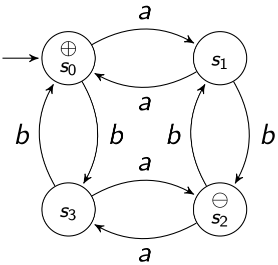

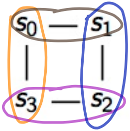

4 maximal cliques

The pseudo-subset construction produces DFAs with 4 states

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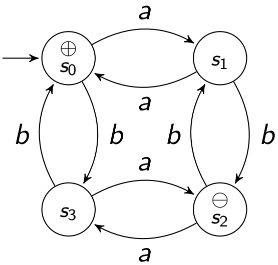

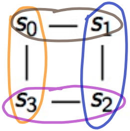

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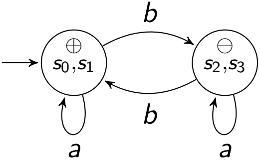
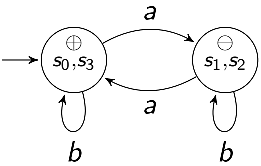
However we can do better, using coverings with two cliques!





Size Bounds of Smallest Compatible DFAs

Theorem 2

Let A be a dcNFA:

- *There exists a compatible DFA whose number of states is bounded by the number of maximal cliques in the compatibility graph of A*

  Upper bound:
  Number of maximal cliques in the compatibility graph

- *Each DFA compatible with A should have at least as many states as the smallest number of cliques covering the compatibility graph of A*

  Lower bound:
  Minimum number of cliques covering the compatibility graph

State Complexity

State Complexity in the General Case

Theorem 3

For each n-state dcNFA ($n \geq 2$) there exists a compatible DFA with at most $f(n)$ states, s.t.

$$f(n) = \begin{cases} 3^{\lfloor n/3 \rfloor}, & \text{if } n \equiv 0 \pmod 3, \\ 4 \cdot 3^{\lfloor n/3 \rfloor - 1}, & \text{if } n \equiv 1 \pmod 3, \\ 2 \cdot 3^{\lfloor n/3 \rfloor}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Furthermore this bound can be effectively reached

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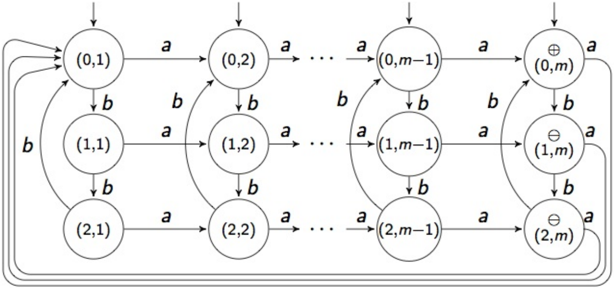
Proof

Upper bound:
$f(n)$ is the maximal number of maximal cliques in a graph with n vertices (Moon & Moser, 1965)

Proof

Lower bound:
from the lower bound for the conversion of *self-verifying automata* into DFAs (Jirásková & Pighizzini, 2011)



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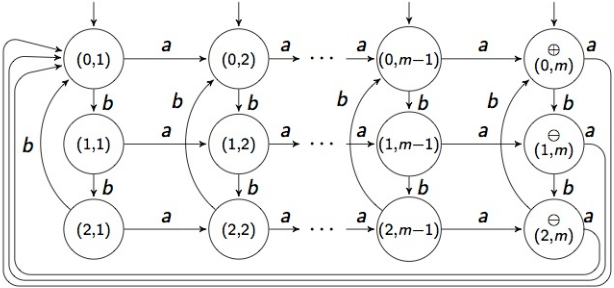

With a *single initial state* (but *nondeterministic transitions*), the optimal state bound remains the same

State Complexity in the Deterministic Case

Let A be an n-state dcDFA

- There exists a compatible DFA with n states
  which is obtained by arbitrarily marking each don't care state
  either as accepting or as rejecting

- This bound cannot be reduced

  Worst case:

  A does not contain any *don't care* state and it is minimal

Time Complexity

NP-completness

Theorem 4

The following problem is NP-complete:

> *Given a dcNFA A and an integer $k > 0$,*
> *does there exist a compatible DFA with $\leq k$ states?*

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In polynomial time we can

- nondeterministically generate a DFA B with $\leq k$ states
- verify that B is compatible with A, as follows:

  for each reachable state (p, q) in the "product" of A and B the following conditions should be verified
  - if p accepting in A then q is final in B
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NP-hardness follows from (Pfleeger, 1973)
(even if A is a dcDFA!) □

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Minimization of dcDFAs and dcNFAs is NP-complete

Conclusion

Our Contributions

- Characterization of DFAs compatible with each given dcNFA
- Pseudo-subset construction
- Upper and lower bounds for the number of states of the smallest compatible DFA
- NP-completeness of the reduction of dcDFAs and dcNFAs to minimal compatible DFAs
- dcNFAs over a one-letter alphabet

Some Possible Future Investigations

- Classes of dcNFAs with compatible DFAs of polynomial size
- Operations on dcNFAs and their state complexity
- Extension of *don't care* notion to other devices
- ...

Thank you for your attention!