

Two-Way Automata and Descriptive Complexity

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What is Descriptive Complexity?

A Classical Example: NFAs vs DFAs

Formal language point of view:

- ▶ The class of languages recognized by NFAs coincides with the class of languages recognized by DFAs

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Descriptive complexity point of view:

- ▶ Each n -state NFA can be simulated by a DFA *with 2^n states* [Rabin&Scott '59]
- ▶ For each integer n there exists a language L_n s.t.:
 - ▶ L_n is accepted by an n -state NFA
 - ▶ the minimum DFA for L_n requires 2^n states[Meyer&Fischer '71]
- ▶ Hence:

The exact cost, in terms of states, of the simulation of NFAs by DFAs is 2^n

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Descriptive Complexity

Given

\mathcal{C} a class of languages

\mathcal{S} a formal system (e.g., class of devices, class of grammars,...)
able to represent all the languages in \mathcal{C}

What is the *size* of the representations of the languages in \mathcal{C}
by the system \mathcal{S} ?

Descriptive complexity compares different descriptions
of a same class of languages:

\mathcal{S}' another formal system able to represent all the languages in \mathcal{C} :

Question

*Find the relationships between the sizes of the representations in
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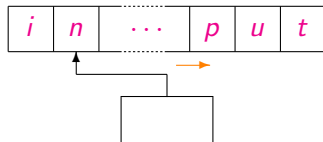
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One-way and Two-way Automata

Finite State Automata



One-way version

At each step the input head is moved
one position to the right

- ▶ 1DFA: *deterministic* transitions
- ▶ 1NFA: *nondeterministic* transitions

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$$H_n = (a + b)^{n-1} a (a + b)^*$$

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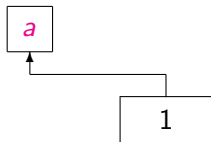
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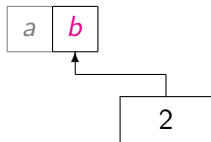
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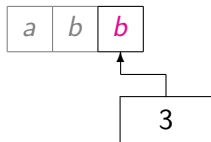
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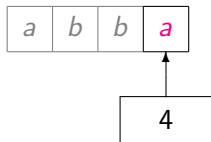
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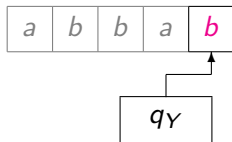
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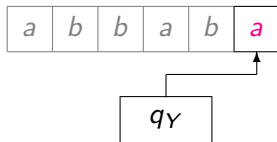
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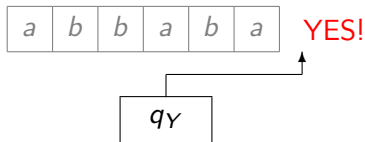
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1DFA: $n + 2$ states

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How to locate it?

Use nondeterminism!

Guess Reading the symbol a the automaton can guess that it is the n th symbol from the right

Verify In the next steps the automaton verifies such a guess

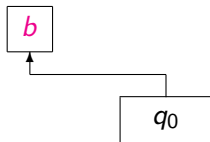
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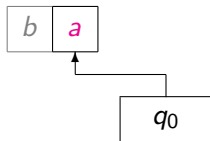
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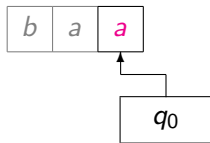
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guess

4th symbol from the right

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Check the n th symbol from the right!

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verify

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verify

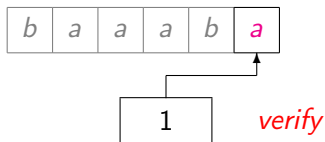
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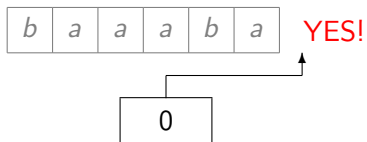
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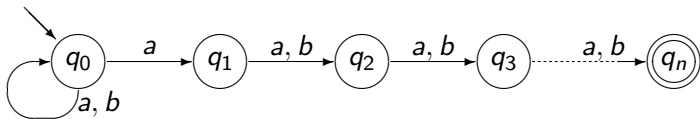
1NFA: $n + 1$ states

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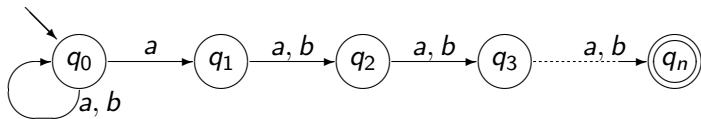


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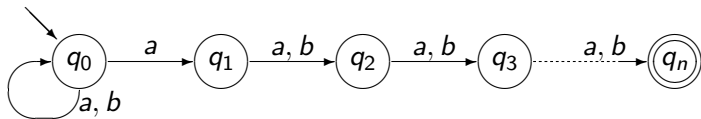
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Very nice!

...but I need a *deterministic* automaton...

Remember the previous n input symbols!

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Ex. $n = 4$

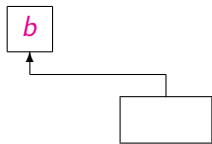
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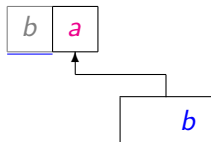
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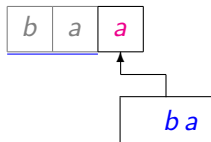
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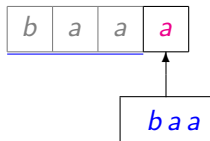
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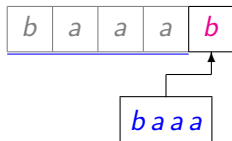
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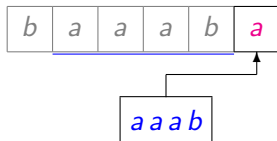
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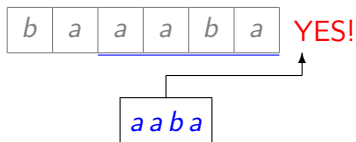
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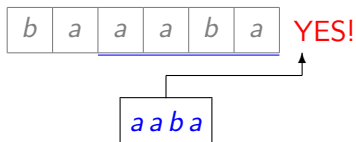
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1DFA: 2^n states

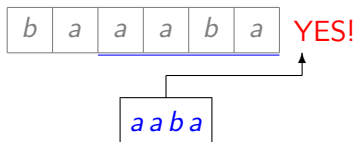
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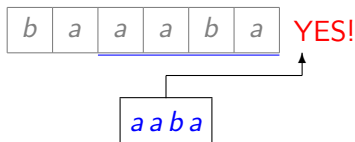
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1DFA: 2^n states

...but I need a smaller deterministic automaton...

This is the smallest one!

However...

A Preliminary Example

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Check the n th symbol from the right!

...if the head can be moved back...

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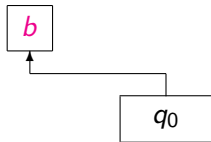
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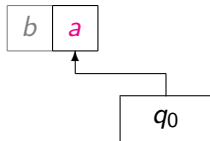
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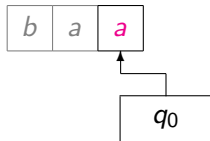
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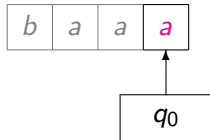
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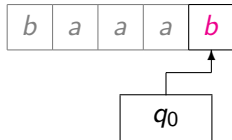
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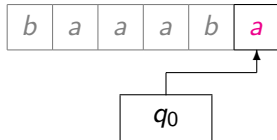
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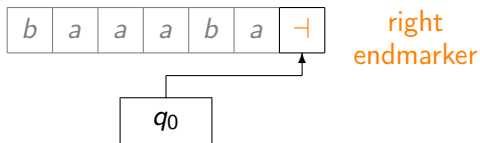
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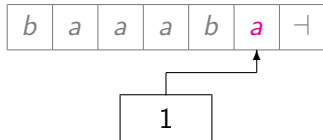
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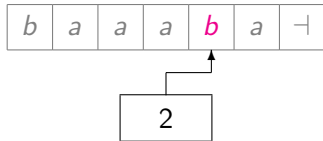
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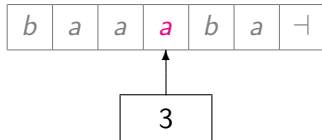
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decision

**if input symbol = a then accept
else reject**

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YES!

4

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Two-way deterministic automaton (2DFA): $n + \dots$ states

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Check the n th symbol from the right!

Summing up, I_n is accepted by

- ▶ a 1NFA and a 2DFA with approximately the same number of states $n + \dots$
- ▶ each 1DFA is exponentially larger ($\geq 2^n$ states)

A Preliminary Example

$\Sigma = \{a, b\}$, fixed $n > 0$:

$$I_n = (a + b)^* a (a + b)^{n-1}$$

Check the n th symbol from the right!

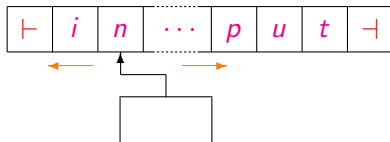
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In this example,

nondeterminism can be removed using two-way motion
keeping approximately the same number of states

Two-Way Automata: Technical Details



- ▶ Input surrounded by the *endmarkers* \vdash and \dashv
- ▶ Moves
 - to the *left*
 - to the *right*
 - *stationary*
- ▶ Initial configuration
- ▶ Accepting configuration
- ▶ Infinite computations are possible
- ▶ *Deterministic* (2DFA) and *nondeterministic* (2NFA) versions

1DFA, 1NFA, 2DFA, 2NFA

What about the power of these models?

1DFA, 1NFA, 2DFA, 2NFA

What about the power of these models?

They share the same computational power, namely they characterize the class of *regular languages*,

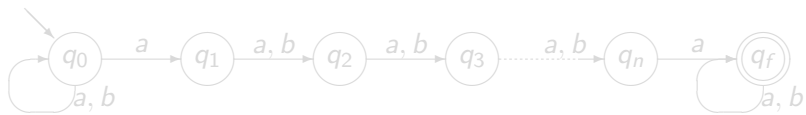
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What about the power of these models?

They share the same computational power, namely they characterize the class of *regular languages*, **however...**

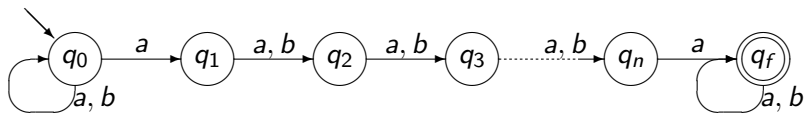
...some of them are more succinct

Main Example: $L_n = (a + b)^* a (a + b)^{n-1} a (a + b)^*$



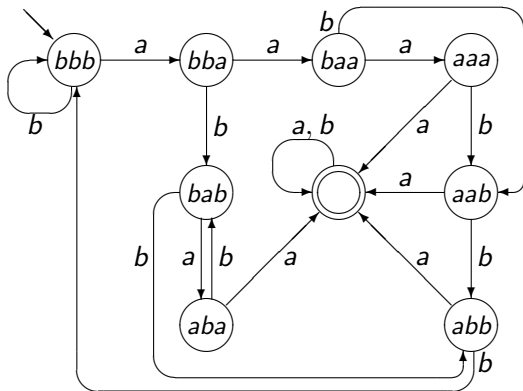
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1NFA: $n + 2$ states

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$n = 3$

Minimum 1DFA: $2^n + 1$ states

Main Example: $L_n = (a + b)^* a (a + b)^{n-1} a (a + b)^*$

2DFA ?

Even scanning from the right it seems that we need to remember a “window” of n symbols

We use a different technique!

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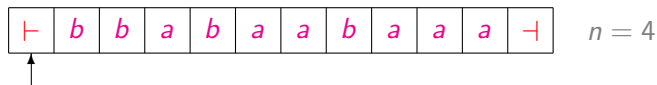
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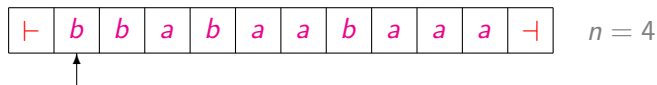
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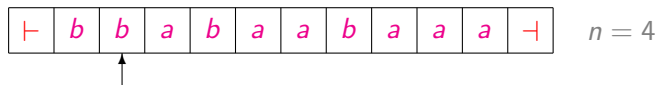
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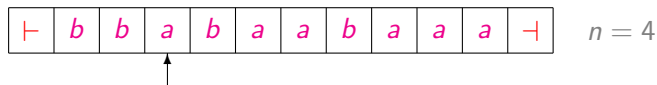
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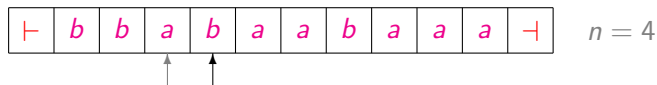
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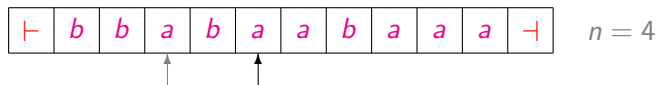
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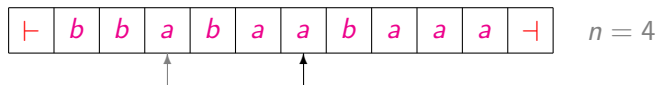
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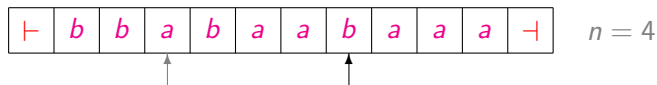
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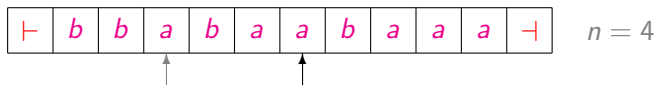
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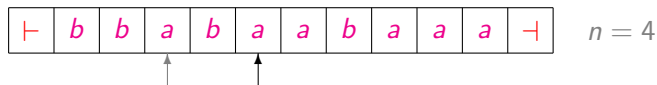
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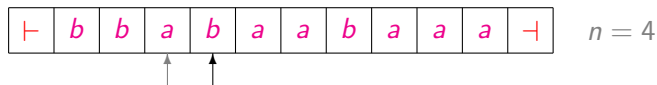
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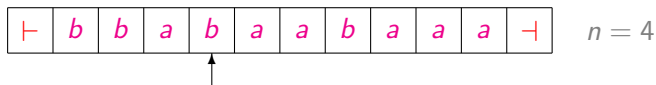
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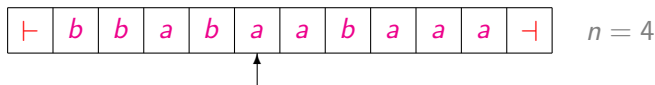
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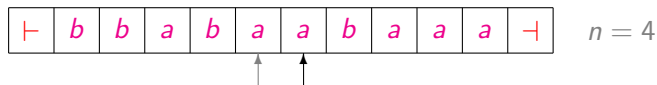
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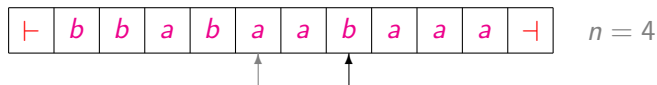
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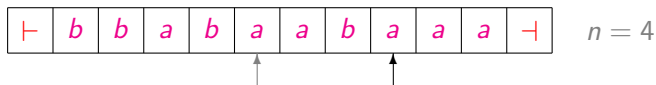
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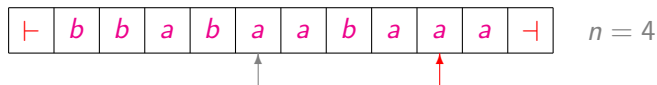
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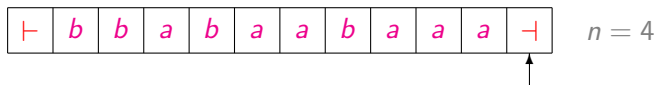
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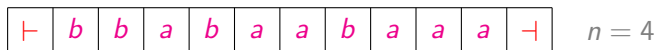
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2DFA: $2n + \dots$ states

Main Example: $L_n = (a + b)^* a (a + b)^{n-1} a (a + b)^*$

Summing up,

- ▶ L_n is accepted by
 - a 1NFA
 - a 2DFAwith $O(n)$ states
- ▶ Each 1DFA is exponentially larger

Also for this example,
nondeterminism can be removed using two-way motion
keeping a linear number of states

Is it always possible
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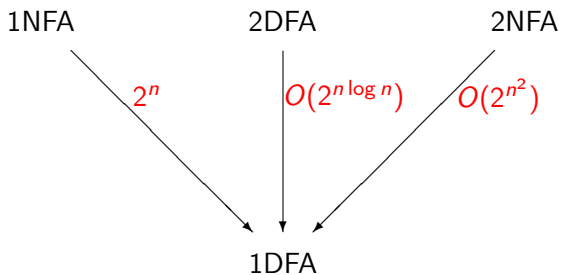
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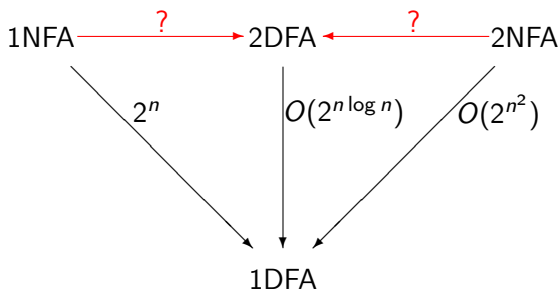
The Question of Sakoda and Sipser

Costs of the Optimal Simulations Between Automata



[Rabin&Scott '59, Shepardson '59, Meyer&Fischer '71, ...]

Costs of the Optimal Simulations Between Automata

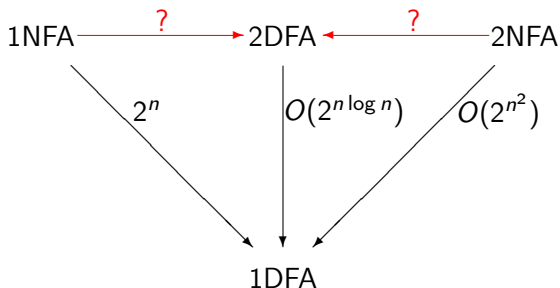


Problem ([Sakoda&Sipser '78])

Do there exist polynomial simulations of

- ▶ *1NFAs by 2DFAs*
- ▶ *2NFAs by 2DFAs ?*

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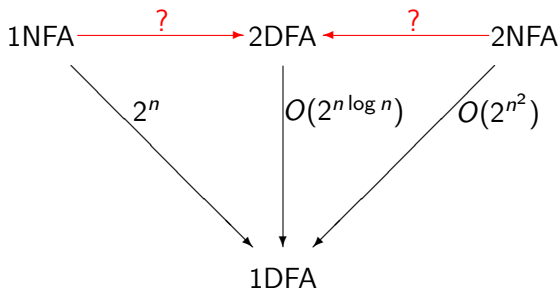
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- ▶ 1NFAs by 2DFAs
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Conjecture

These simulations are not polynomial

Costs of the Optimal Simulations Between Automata



- ▶ **Exponential upper bounds**
deriving from the simulations of 1NFAs and 2NFAs by 1DFAs
- ▶ **Polynomial lower bound**
 $\Omega(n^2)$ for the cost of the simulation of 1NFAs by 2DFAs

[Chrobak '86]

Sakoda and Sipser Question

- ▶ Very difficult in its general form
- ▶ Not very encouraging obtained results:
 - Lower and upper bounds too far
(Polynomial vs exponential)
- ▶ Hence:
 - Try to attack restricted versions of the problem!

NFAs vs 2DFAs: Restricted Versions

(i) Restrictions on the resulting machines (2DFAs)

- ▶ sweeping automata [Sipser '80]
- ▶ oblivious automata [Hromkovič&Schnitger '03]
- ▶ “few reversal” automata [Kapoutsis '11]

(ii) Restrictions on the languages

- ▶ unary regular languages [Geffert Mereghetti&P '03]

(iii) Restrictions on the starting machines (2NFAs)

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Restrictions on the resulting machines

Restricted Models: Separations

oblivious


sweeping

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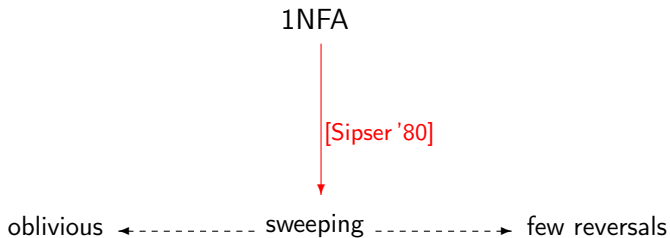
 $O(n^2)$

Restricted Models: Separations

oblivious ←----- sweeping -----→ few reversals

$O(n^2)$ →

Restricted Models: Separations



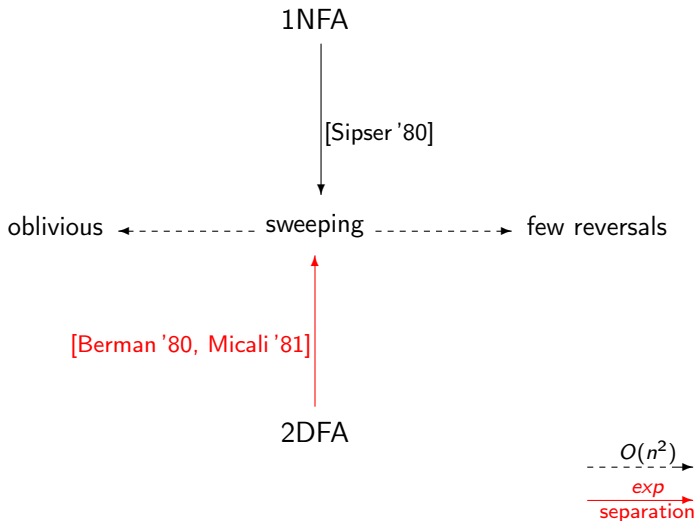
$O(n^2)$

exp

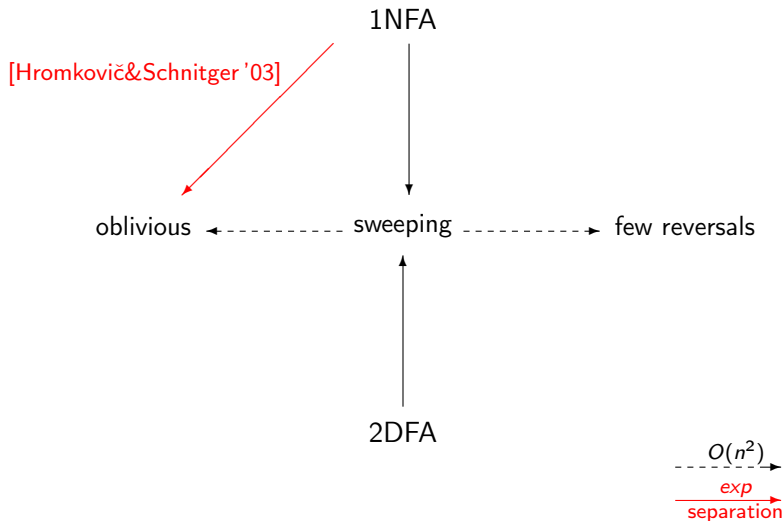
separation

The diagram shows two horizontal arrows pointing to the right. The top arrow is dashed and labeled with the complexity $O(n^2)$. The bottom arrow is solid red and labeled with 'exp' in red above it and 'separation' in red below it.

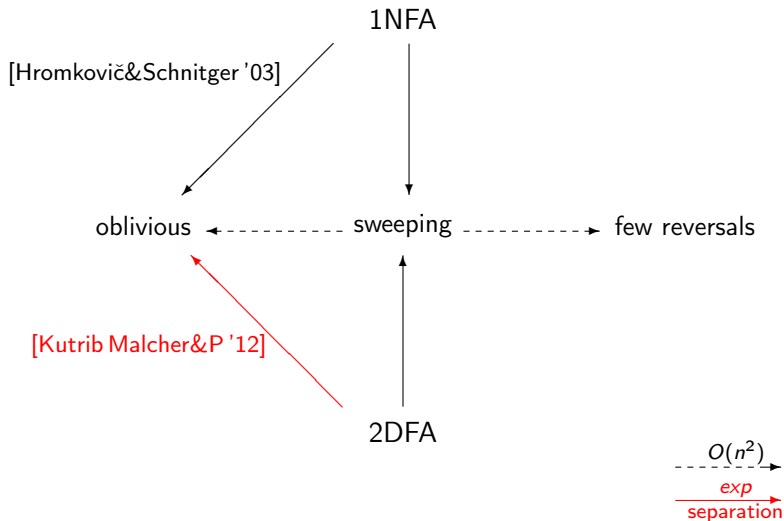
Restricted Models: Separations



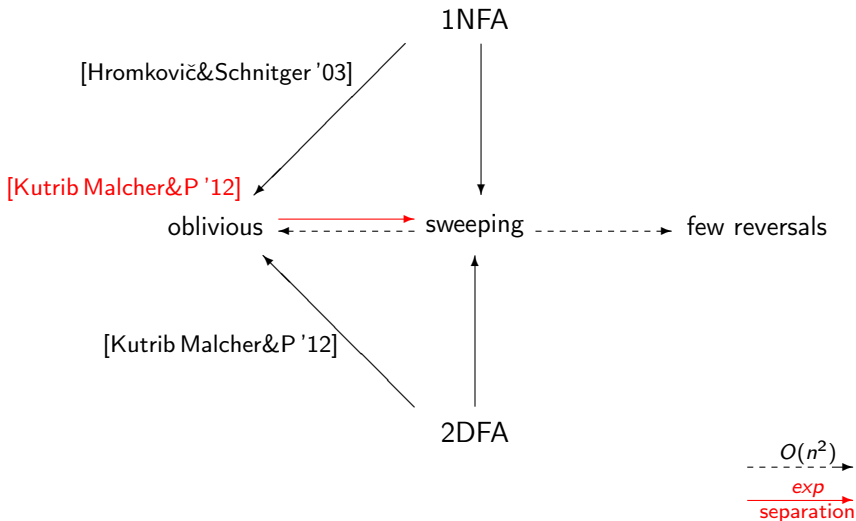
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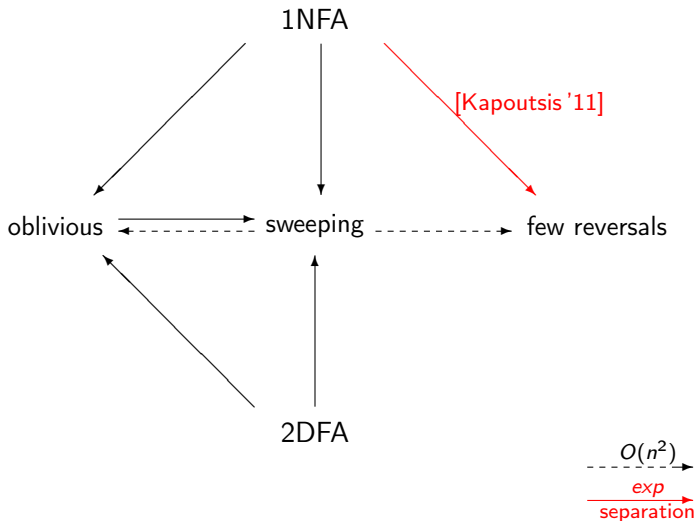
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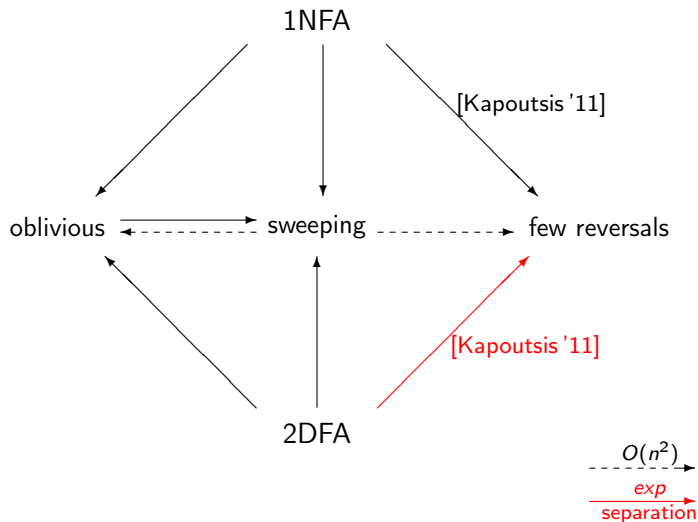
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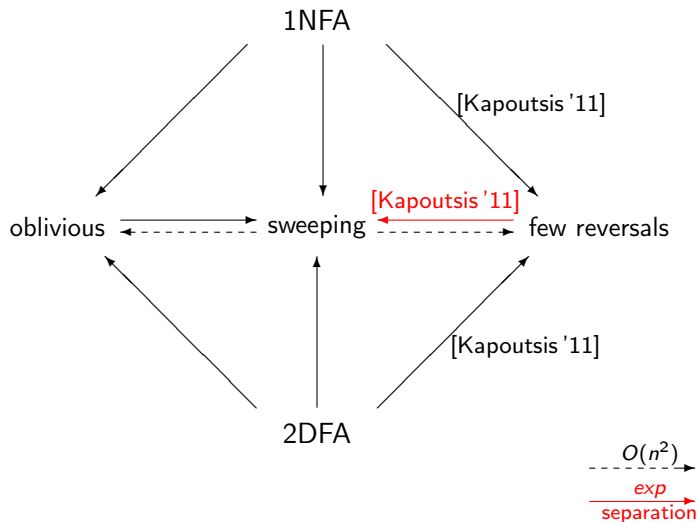
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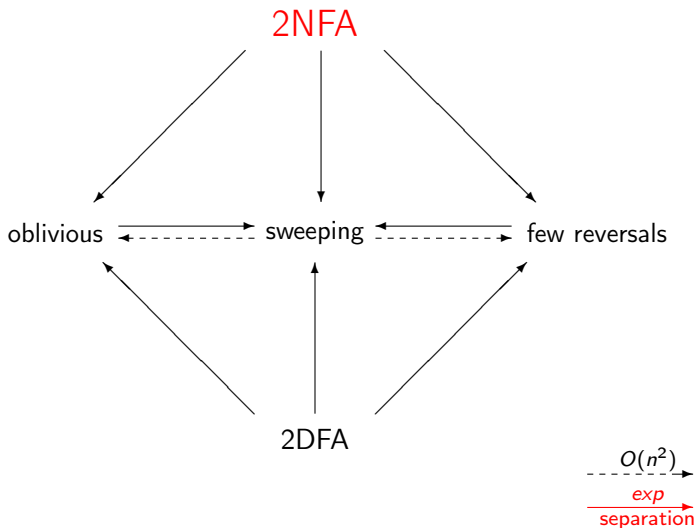
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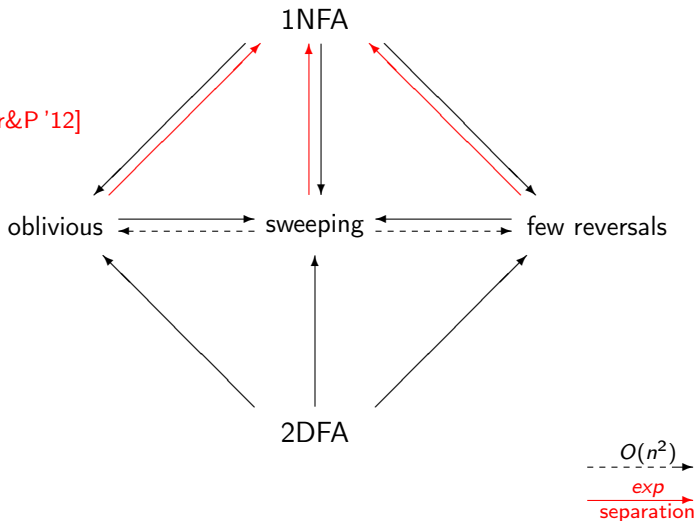


Restricted Models: Separations



Restricted Models: Separations

[Kutrib Malcher&P '12]



The Unary Case

$$\#\Sigma = 1$$

Optimal Simulation Between Unary Automata

The costs of the optimal simulations between automata are different in the unary and in the general case

1DFA

1NFA

2DFA

2NFA

Optimal Simulation Between Unary Automata

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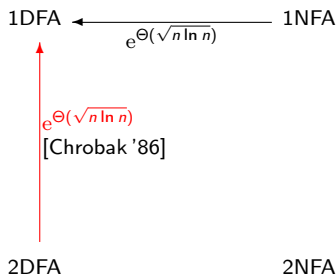
1DFA $\xleftarrow[\text{Chrobak '86}]{e^{\Theta(\sqrt{n \ln n})}}$ 1NFA

2DFA

2NFA

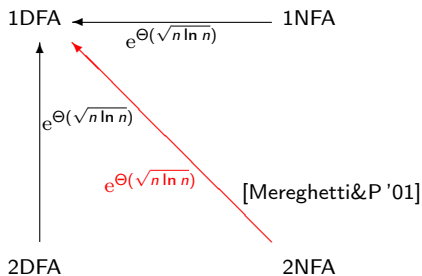
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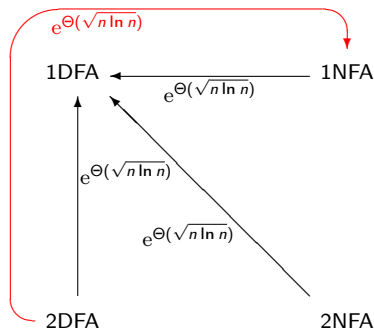
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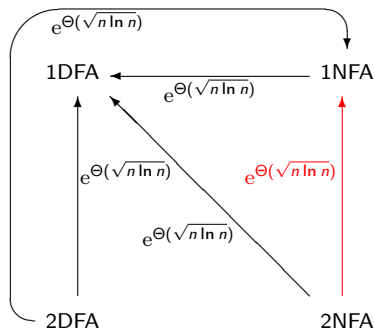
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follows from 2DFA \rightarrow 1DFA

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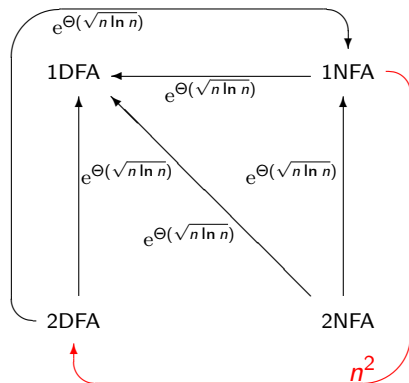
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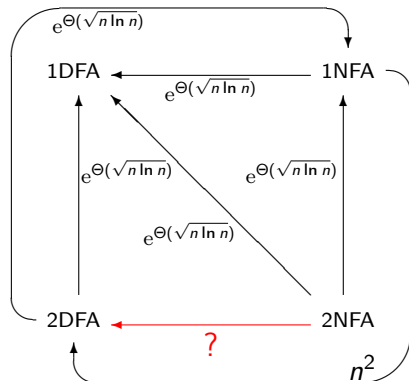


1NFA \rightarrow 2DFA
In the unary case
this question is solved!
(polynomial conversion)

[Chrobak '86]

Optimal Simulation Between Unary Automata

The costs of the optimal simulations between automata are different in the unary and in the general case



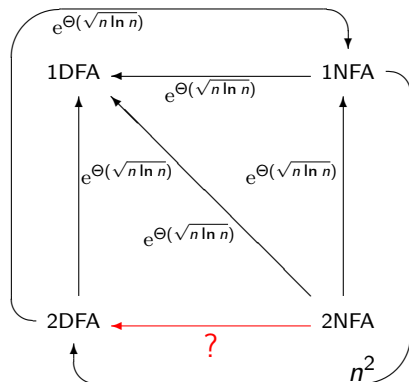
2NFA \rightarrow 2DFA

Even in the unary case
this question is open!

- ▶ $e^{\Theta(\sqrt{n \ln n})}$ upper bound
(from 2NFA \rightarrow 1DFA)
- ▶ $\Omega(n^2)$ lower bound
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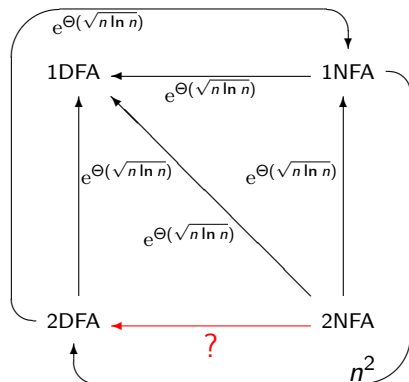
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- ▶ $\Omega(n^2)$ lower bound
(from 1NFA \rightarrow 2DFA)

A better upper bound $e^{O(\ln^2 n)}$
has been proved!

A Normal Form for Unary 2NFAs

Theorem ([Geffert Mereghetti&P '03])

For each unary n -state 2NFA A there exists an almost equivalent 2NFA M s.t.

- ▶ *M makes nondeterministic choices and changes the head direction only visiting the end-markers*
- ▶ *M has $N \leq 2n + 2$ many states*
- ▶ *$L(A)$ and $L(M)$ can differ only on strings of length $\leq 5n^2$*

Normal Form for Unary 2NFAs: Consequences

(i) Subexponential simulation of unary 2NFAs by 2DFAs

Each unary n -state 2NFA can be simulated by a 2DFA
with $e^{O(\ln^2 n)}$ states [Geffert Mereghetti&P '03]

(ii) Polynomial complementation of unary 2NFAs

Inductive counting argument [Geffert Mereghetti&P '07]

(iii) Polynomial simulation of unary 2NFAs by 2DFAs

under the condition $L = NL$ [Geffert&P '11]

Hence, proving that the upper bound $e^{O(\ln^2 n)}$ is tight
would separate L and NL in the *general case*

(iv) Polynomial simulation of unary 2NFAs by unambiguous 2NFAs

(unconditional) [Geffert&P '11]

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Restricted 2NFAs

Outer Nondeterministic Automata (OFAs) [Guillon Geffert&P '12]:

- ▶ *nondeterministic choices*

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- ▶ *nondeterministic choices*

are *possible only* when the head is visiting the *endmarkers*

Hence:

- ▶ No restrictions on the *input alphabet*
- ▶ No restrictions on *head reversals*
- ▶ *Deterministic transitions* on “real” input symbols

Outer Nondeterministic Automata (OFAs)

The results we obtained for the unary case
can be extended to 2OFAs:

[Guillon Geffert&P '12]

- (i) Subexponential simulation of 2OFAs by 2DFAs
- (ii) Polynomial complementation of 2OFAs
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While in the unary case all the proofs rely
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Sakoda&Sipser Question: Current Knowledge

► Upper bounds

| | 1NFA→2DFA | 2NFA→2DFA |
|---------------------------|---------------------|------------------|
| unary case and OFAs | $O(n^2)$ optimal | $e^{O(\ln^2 n)}$ |
| general case | exponential | exponential |

Unary case [Chrobak '86, Geffert Mereghetti&P '03]

OFAs [Guillon Geffert&P '12]

► Lower Bounds

In all the cases, the best known lower bound is $\Omega(n^2)$

[Chrobak '86]

Conclusion

Final Remarks

- ▶ The question of Sakoda and Sipser is very challenging
- ▶ In the investigation of restricted versions many interesting and not artificial models have been considered
- ▶ The results obtained under restrictions, even if not solving the full problem, are not trivial and, in many cases, very deep
- ▶ Connections with space and structural complexity
 - questions
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Thank you for your attention!