

Parikh Equivalence and Descriptive Complexity

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(SOFSEM 2012, DLT 2012, Inf. and Comput. 2013)



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DI MILANO

NFAs vs DFAs

Subset construction: [Rabin&Scott '59]

NFA
 n states



DFA
 2^n states

The state bound cannot be reduced

[Lupanov '63, Meyer&Fischer '71, Moore '71]

What happens if we do not care of the order of symbols in the strings?

This problem is related to the concept of *Parikh Equivalence*

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This problem is related to the concept of *Parikh Equivalence*

Parikh Equivalence

- ▶ $\Sigma = \{a_1, \dots, a_m\}$ alphabet of m symbols
- ▶ Parikh's map $\psi : \Sigma^* \rightarrow \mathbb{N}^m$:

$$\psi(w) = (|w|_{a_1}, |w|_{a_2}, \dots, |w|_{a_m})$$

for each string $w \in \Sigma^*$

- ▶ Parikh's image of a language $L \subseteq \Sigma^*$:

$$\psi(L) = \{\psi(w) \mid w \in L\}$$

- ▶ $w' =_{\pi} w''$ iff $\psi(w') = \psi(w'')$
- ▶ $L' =_{\pi} L''$ iff $\psi(L') = \psi(L'')$

Parikh's Theorem

Theorem ([Parikh '66])

The Parikh image of a context-free language is a semilinear set, i.e, each context-free language is Parikh equivalent to a regular language

Example:

- ▶ $L = \{a^n b^n \mid n \geq 0\}$
 - ▶ $R = (ab)^*$
- $$\psi(L) = \psi(R) = \{(n, n) \mid n \geq 0\}$$

Different proofs after the original one of Parikh, e.g.

- ▶ [Goldstine '77]: a simplified proof
- ▶ [Aceto&Ésik&Ingólfssdóttir '02]: an equational proof
- ▶ ...
- ▶ [Esparza&Ganty&Kiefer&Luttenberger '11]: complexity aspects

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Our Goal

We want to convert nondeterministic automata and context-free grammars into *small Parikh equivalent* deterministic automata

Problem (NFAs to DFAs)

NFA
n states

\Longrightarrow_{π}

DFA
how many states?

Problem (CFGs to DFAs)

CFG
size n

\Longrightarrow_{π}

DFA
how many states?

Why?

- ▶ Interesting theoretical properties:
wrt Parikh equivalence regular and context-free languages are indistinguishable [Parikh '66]
- ▶ Connections of with:
 - Semilinear sets
 - Presburger Arithmetics [Ginsburg&Spanier '66]
 - Petri Nets [Esparza '97]
 - Logical formulas [Verma&Seidl&Schwentick '05]
 - Formal verification [Dang&Ibarra&Bultan&Kemmerer&Su'00, Göller&Mayr&To'09]
 - ...
- ▶ Unary case:
size costs of the simulations of CFGs and PDAs by DFAs [Pighizzini&Shallit&Wang '02]

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Converting NFAs

Problem (NFAs to DFAs)

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\Rightarrow_{π}

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how many states?

- ▶ *Upper bound*

Subset construction: 2^n

- ▶ *Lower bound*

Conversion *NFAs* \rightarrow *DFAs* in the unary case: $e^{\Theta(\sqrt{n \ln n})}$

[Chrobak '86]

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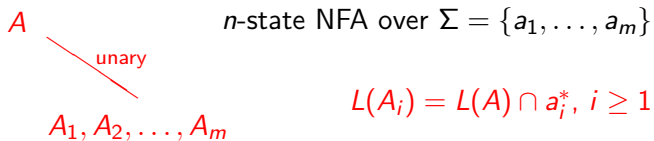
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Converting NFAs: General Idea

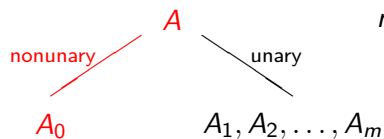
A n -state NFA over $\Sigma = \{a_1, \dots, a_m\}$

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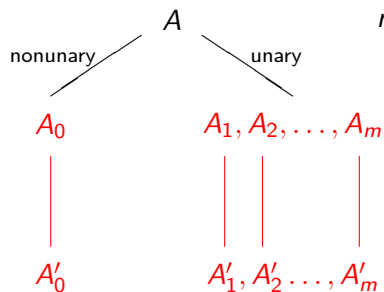


$$L(A_i) = L(A) \cap a_i^*, i \geq 1$$

$$L(A_0) = L - \bigcup_{i=1}^m L(A_i)$$

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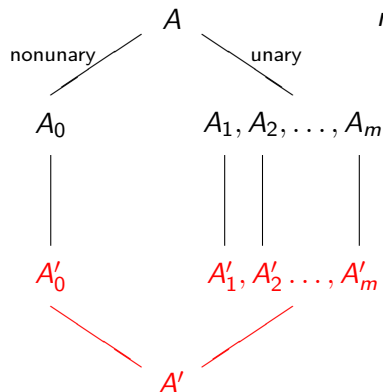
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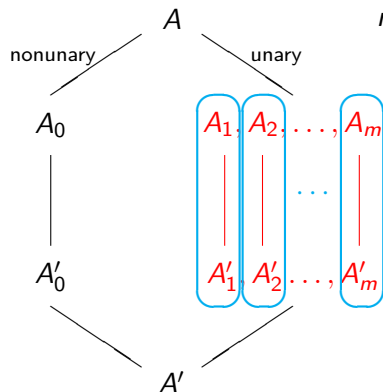
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Parikh equivalent DFAs

DFA Parikh equivalent to A

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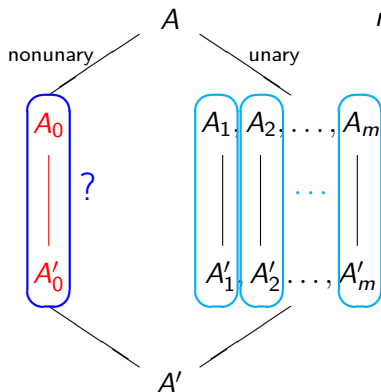
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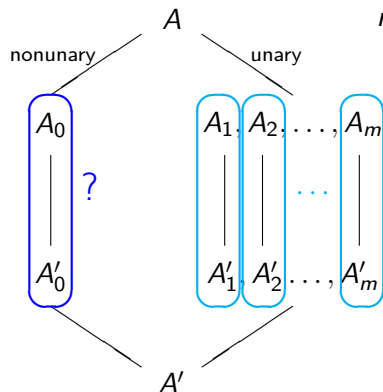
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How much is the state cost of the conversion of NFAs accepting *only nonunary strings* into Parikh equivalent DFAs?

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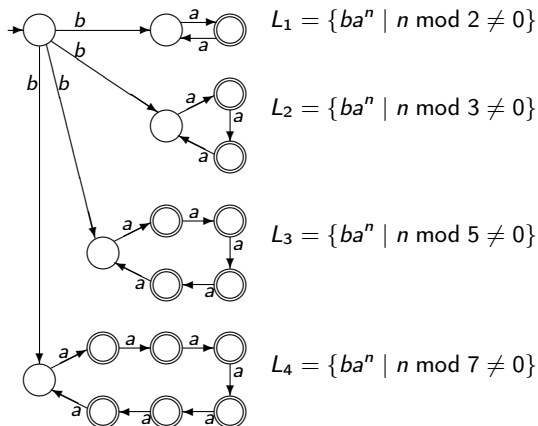
Only polynomial!
(less than in unary case)

An Example

$$L = \{ba^n \mid n \bmod 210 \neq 0\}$$

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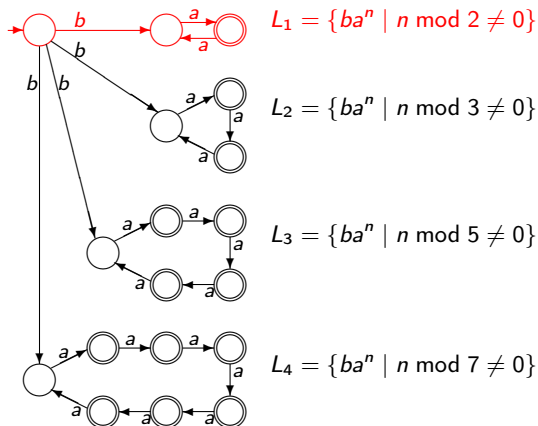
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DFA \geq 211 states

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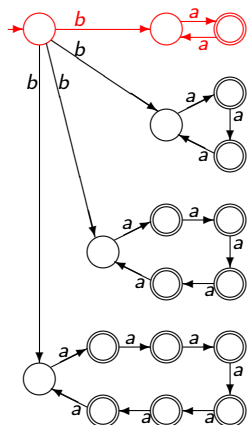
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$$L'_1 = \{ba^n \mid n \bmod 2 \neq 0\}$$

$$L_2 = \{ba^n \mid n \bmod 3 \neq 0\}$$

$$L_3 = \{ba^n \mid n \bmod 5 \neq 0\}$$

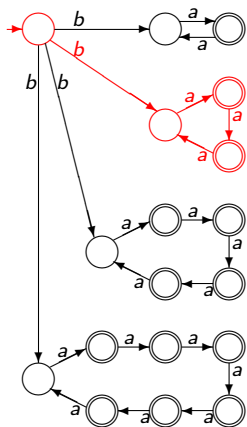
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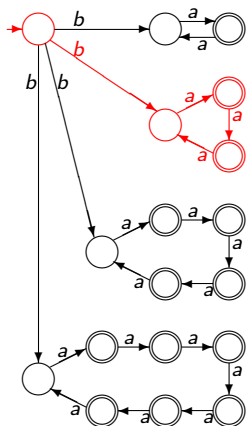
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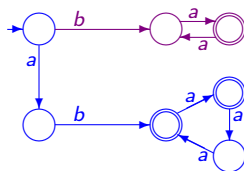
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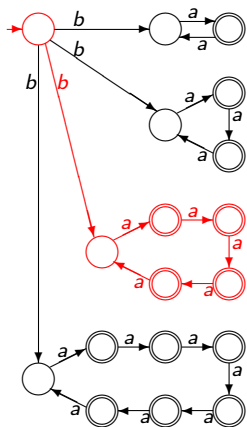
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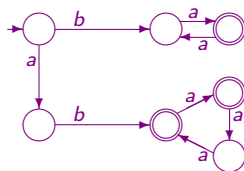
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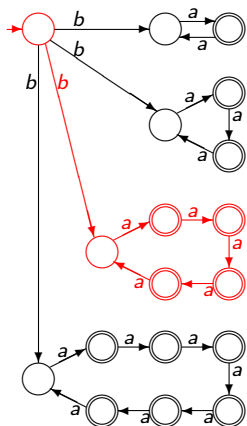
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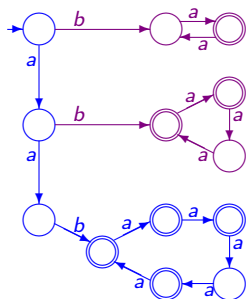
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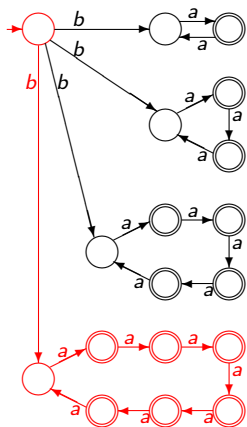
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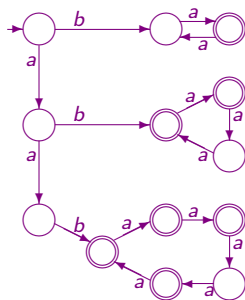
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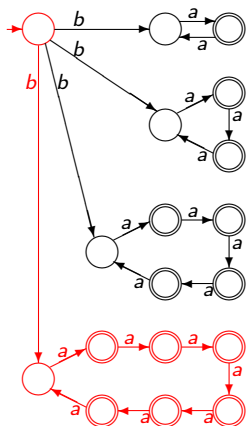
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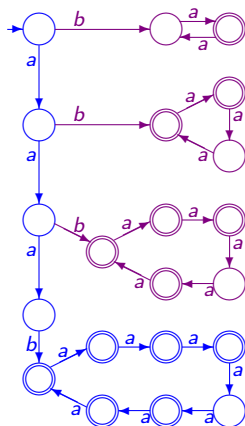
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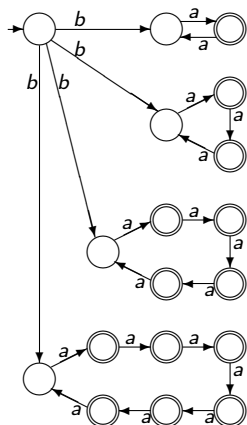
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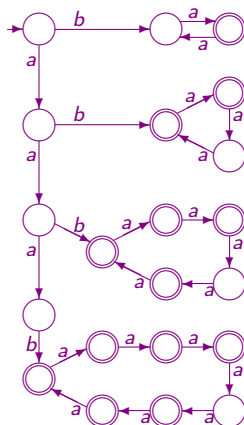
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$$L' = L'_1 \cup L'_2 \cup L'_3 \cup L'_4$$

DFA with only 21 states!

Converting NFAs Accepting Only Nonunary Strings

The conversion uses a modification of the following result:

Theorem ([Kopczyński&To '10])

Given $\Sigma = \{a_1, \dots, a_m\}$, there is a polynomial p s.t. for each n -state NFA A over Σ ,

$$\psi(L(A)) = \bigcup_{i \in I} Z_i$$

where:

- ▶ I is a set of at most $p(n)$ indices
- ▶ for $i \in I$, $Z_i \subseteq \mathbb{N}^m$ is a linear set of the form:

$$Z_i = \{\alpha_0 + n_1\alpha_1 + \dots + n_k\alpha_k \mid n_1, \dots, n_k \in \mathbb{N}\}$$

with

- ▶ $0 \leq k \leq m$
- ▶ the components of α_0 are bounded by $p(n)$
- ▶ $\alpha_1, \dots, \alpha_k$ are linearly independent vectors from $\{0, 1, \dots, n\}^m$

Converting NFAs Accepting Only Nonunary Strings

Outline: linear sets

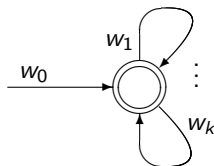
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can be converted into a poly size DFA accepting a language

$$R_i = w_0(w_1 + \cdots + w_k)^*$$

s.t. $\psi(w_j) = \alpha_j$, $j = 0, \dots, k$, and
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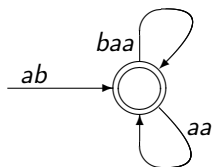
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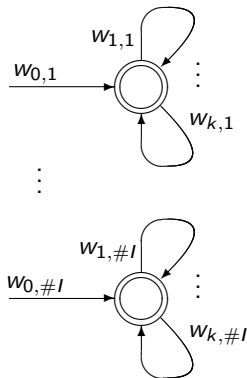
Example:

- ▶ $\{(1, 1) + n_1(2, 1) + n_2(2, 0) \mid n_1, n_2 \geq 0\}$
- ▶ $ab(baa + aa)^*$



Converting NFAs Accepting Only Nonunary Strings

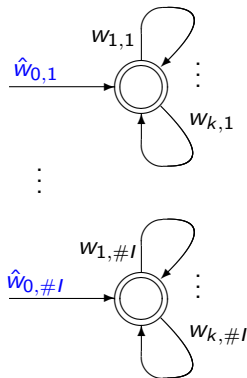
Outline: from linear to semilinear



- ▶ Standard construction for union of DFAs:
number of states = *product*
 $\#I \leq p(n) \Rightarrow$ Too large!!!
- ▶ Strings $w_{0,i}$ can be replaced by Parikh equivalent strings $\hat{w}_{0,i}$ in such a way that $W_0 = \{\hat{w}_{0,i} \mid i \in I\}$ is a *prefix code*
- ▶ After this change:
number of states \leq *sum* Polynomial!!!

Converting NFAs Accepting Only Nonunary Strings

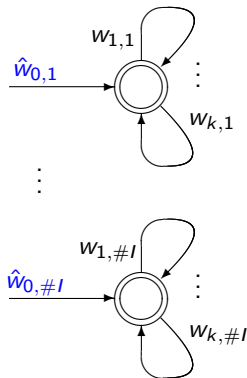
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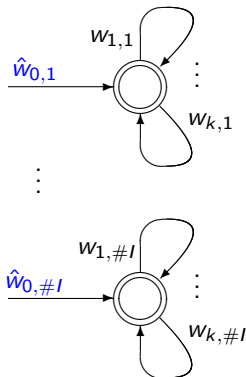
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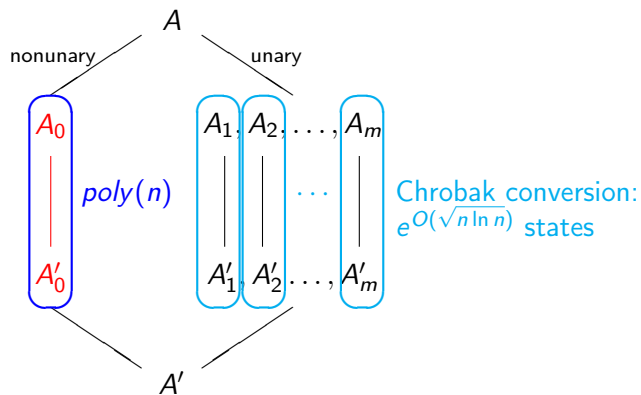


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 $\#I \leq p(n) \Rightarrow$ Too large!!!
- ▶ Strings $w_{0,i}$ can be replaced by Parikh equivalent strings $\hat{w}_{0,i}$ in such a way that $W_0 = \{\hat{w}_{0,i} \mid i \in I\}$ is a *prefix code*
- ▶ After this change:
number of states \leq *sum* Polynomial!!!

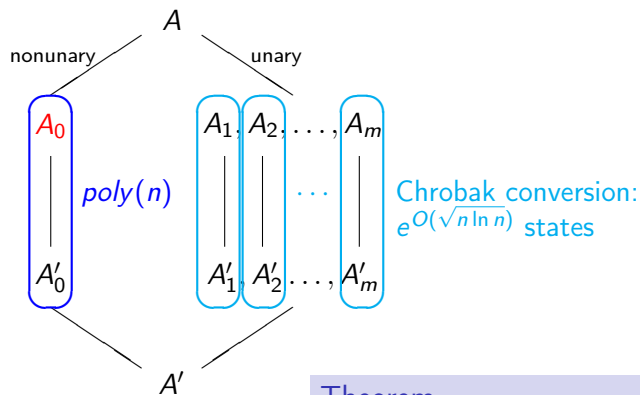
Theorem

For each n -state NFA accepting a language none of whose words are unary, there exists a Parikh equivalent DFA with a number of states polynomial in n

Converting NFAs: Back to the General Case



Converting NFAs: Back to the General Case



Theorem

For each n -state NFA there exists a Parikh equivalent DFA with $e^{O(\sqrt{n \ln n})}$ states. Furthermore this cost is tight

Converting CFGs

Problem (CFGs to NFAs and DFAs)

CFG
size h

\implies_{π}

NFA/DFA
how many states?

- ▶ We consider CFGs in Chomsky Normal Form
- ▶ As a measure of size we consider the *number of variables*

[Gruska '73]

Converting CFGs into Parikh Equivalent Automata

Conversion into *Nondeterministic Automata*

Problem (CFGs to NFAs)

CFG
Chomsky normal form
h variables

\implies_{π}

NFA
how many states?

Upper bound:

- $2^{2^{O(h^2)}}$ implicit construction from classical proof of Parikh's Th.
- $O(4^h)$ [Esparza&Ganty&Kiefer&Luttenberger '11]

Lower bound: $\Omega(2^h)$

Folklore

Converting CFGs into Parikh Equivalent Automata

Conversion into *Deterministic Automata*

Problem (CFGs to DFAs)

CFG
Chomsky normal form
 h variables

\Rightarrow_{π}

DFA
how many states?

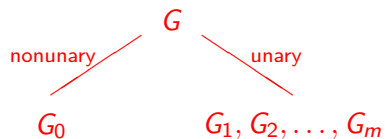
- ▶ Upper bound: $2^{O(4^h)}$ subset construction
- ▶ Lower bound: 2^{ch^2} tight bound for the unary case $2^{\Theta(h^2)}$
[Pighizzini&Shallit&Wang '02]

Converting CFGs into Parikh Equivalent DFAs

G

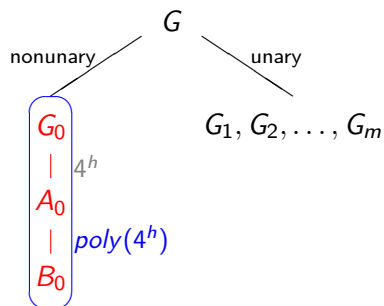
CFG with h variables

Converting CFGs into Parikh Equivalent DFAs



CFG with h variables

Converting CFGs into Parikh Equivalent DFAs

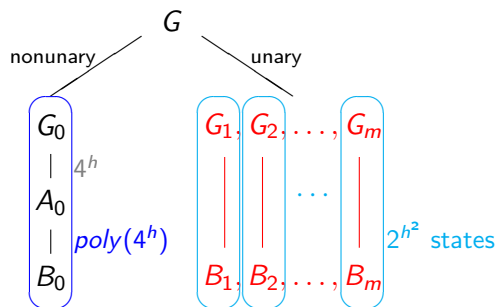


CFG with h variables

Parikh equivalent NFA

Parikh equivalent DFA

Converting CFGs into Parikh Equivalent DFAs

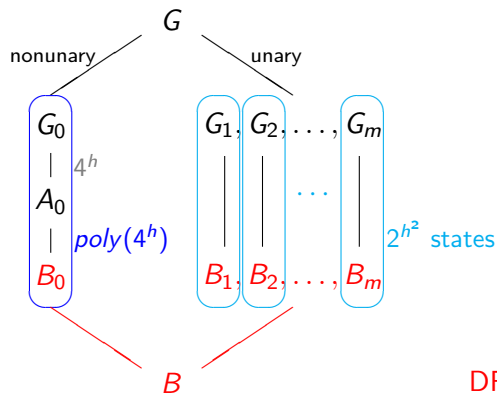


CFG with h variables

Parikh equivalent NFA

Parikh equivalent DFAs

Converting CFGs into Parikh Equivalent DFAs



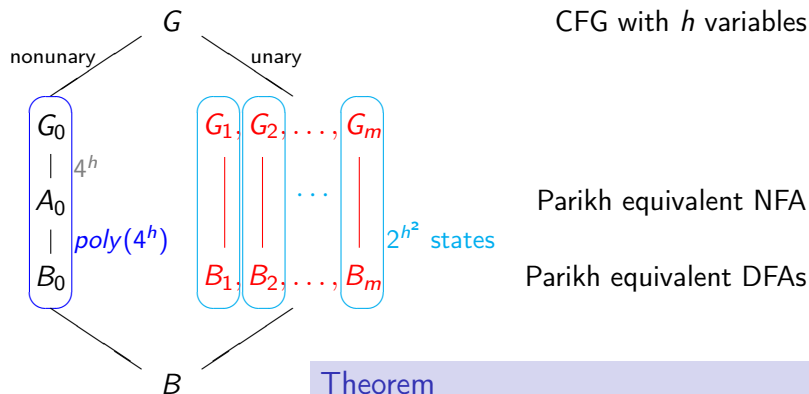
CFG with h variables

Parikh equivalent NFA

Parikh equivalent DFAs

DFA Parikh equivalent to G

Converting CFGs into Parikh Equivalent DFAs



Theorem

For any CFG in Chomsky normal form with h variables, there exists a Parikh equivalent DFA with at most $2^{O(h^2)}$ states. Furthermore this bound is tight

Final Considerations

We obtained the following tight conversions:

	DFA	
NFA n states	$e^{O(\sqrt{n \ln n})}$ states	
CFG Cnf h variables	$2^{O(h^2)}$ states	

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- ▶ It could be interesting to investigate other conversions, e.g., automata minimization under Parkih equivalence, and computational complexity aspects

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Final Considerations

We obtained the following tight conversions:

	DFA	2DFA
NFA n states	$e^{O(\sqrt{n \ln n})}$ states	
CFG Cnf h variables	$2^{O(h^2)}$ states	

What about the conversions into *two-way automata*?

Final Considerations

Conversions into *two-way deterministic automata* (2DFAs)

	DFA	2DFA
NFA n states	$e^{O(\sqrt{n \ln n})}$ states	$poly(n)$ states
CFG Cnf h variables	$2^{O(h^2)}$ states	

Final Considerations

Conversions into *two-way deterministic automata* (2DFAs)

	DFA	2DFA
NFA n states	$e^{O(\sqrt{n \ln n})}$ states	$poly(n)$ states
CFG Cnf h variables	$2^{O(h^2)}$ states	$2^{O(h)}$ states

Thank you for your attention!