Parikh's Theorem and Descriptional Complexity

Giovanna J. Lavado and Giovanni Pighizzini

Dipartimento di Informatica e Comunicazione
Università degli Studi di Milano

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DI MILANO

Parikh's Image

- $\Sigma = \{a_1, \ldots, a_m\}$ alphabet of m symbols
- *Parikh's map* $\psi : \Sigma^* \to \mathbb{N}^m$:

$$\psi(w) = (|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_m})$$

  for each string $w \in \Sigma^*$
- w' and w'' are *Parikh equivalent* iff $\psi(w') = \psi(w'')$ (in symbols $w' =_\pi w''$)
- *Parikh's image* of a language $L \subseteq \Sigma^*$:

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Parikh's Theorem

Theorem ([Parikh '66])

The Parikh image of a context-free language is a semilinear set, i.e,
each context-free language is Parikh equivalent
to a regular language

Example:

- $L = \{a^n b^n \mid n \geq 0\}$
- $R = (ab)^*$

$$\psi(L) = \psi(R) = \{(n, n) \mid n \geq 0\}$$

Different proofs after the original one of Parikh, e.g.

- [Goldstine '77]: a simplified proof
- [Aceto&Ésik&Ingólfsdóttir '02]: an equational proof
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Purpose of the Work

Recent works investigating *complexity aspects* of Parikh's Theorem:

- [Kopczyński&To '10]:
  size of the "semilinear descriptions" of Parikh images of languages defined by NFAs and by CFGs
- [Esparza&Ganty&Kiefer&Luttenberger '11]:
  - new proof of Parikh's Theorem
  - solution to the problem below in the case of nondeterministic automata

Problem

Given a CFG G compare the size of G
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Our aim is to study the same problem for *deterministic automata*

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Why this Problem?

- We came to this problem from the investigation of automata over *a one letter alphabet*
- Costs in states of optimal simulations between different variant unary automata (one-way/two-way, deterministic/nondeterministic) [Chrobak '86, Mereghetti&Pighizzini '01]
- Context-free languages over a unary terminal alphabet are regular [Ginsburg&Rice '62]
- The regularity of unary CFLs is also a corollary of Parikh's Theorem
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Size: Descriptional Complexity Measures

- Finite Automata
  number of states
- Context-Free Grammars
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Unary Context-Free Languages

Theorem ([Pighizzini&Shallit&Wang '02])

For each unary CFG in Chomsky normal form with h variables there are

- *an equivalent NFA with at most $2^{2h-1}+1$ states*
- *an equivalent DFA with less than 2^{h^2} states*

Both bounds are tight

Can we extend this result to larger alphabets?

- The class of CLFs is larger than the class of regular: we cannot have a result of *exactly* the same form!
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Upper and Lower Bounds

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Given a CFG G compare the size of G
with the sizes of finite automata accepting languages
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Nondeterministic automata (number of states wrt s, size of G)

Upper bound:

- $2^{2^{O(s^2)}}$
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Lower bound: $2^{\Omega(2^s)}$

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Is it possible to reduce the gap between
the upper and the lower bound?

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We reduced the upper bound to $2^{s^{O(1)}}$ in the following cases:

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  i.e, context-free subsets of $a_1^* a_2^* \dots a_m^*$ $(m \geq 2)$
- context-free languages over two-letter alphabets

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First Contribution: Bounded Context-Free Languages

Theorem

- $\Sigma = \{a_1, a_2, \ldots, a_m\}$ *fixed alphabet*
- *G grammar in Chomsky normal form with h variables s.t.* $L(G) \subseteq a_1^* a_2^* \ldots a_m^*$

There exists a DFA A with at most $2^{h^{O(1)}}$ states s.t. $L(G) =_\pi L(A)$

First Contribution: Proof Outline

$\Sigma = \{a_1, a_2, \ldots, a_m\}$

- Restriction to *strongly bounded grammars*

  $G = (V, \Sigma, P, S)$ *is strongly bounded iff*
  for all $A \in V$, *there are* $i \leq j$ *s.t.*
  $L_A = \{x \in \Sigma^* \mid A \stackrel{\star}{\Rightarrow} x\} \subseteq a_i^+ a_{i+1}^* \cdots a_{j-1}^* a_j^+$

- $A \in V$ is said to be *unary* iff $L_A \subseteq a_i^+$ for some i

  in this case L_A *is accepted by a DFA with* $< 2^{h^2}$ *states*
  [Pighizzini&Shallit&Wang '02]

- The use of nonunary variables is very restricted:

  If $S \stackrel{\star}{\Rightarrow} \alpha$ then α contains $\leq m - 1$ nonunary variables

  Hence a finite control of size $O(h^{m-1})$ can keep track of them

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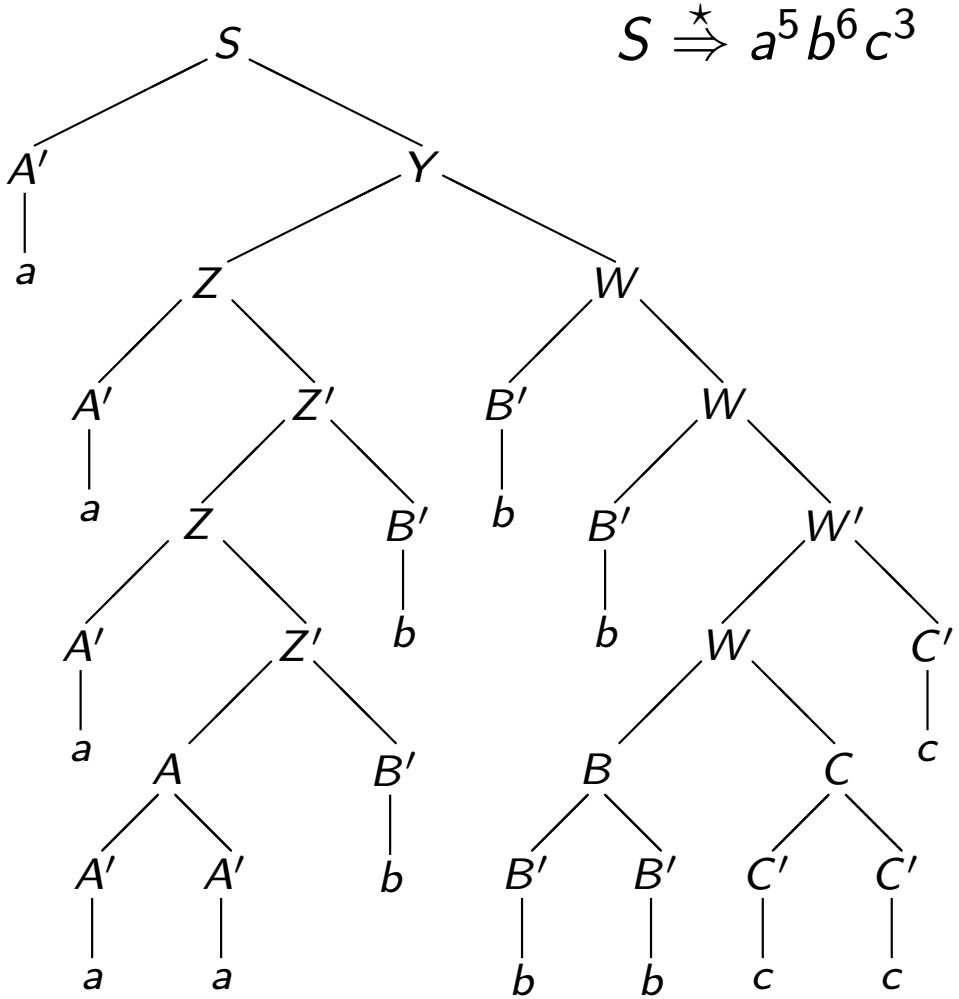
Example $\Sigma = \{a, b, c\}$

$$S \stackrel{\star}{\Rightarrow} a^5 b^6 c^3$$



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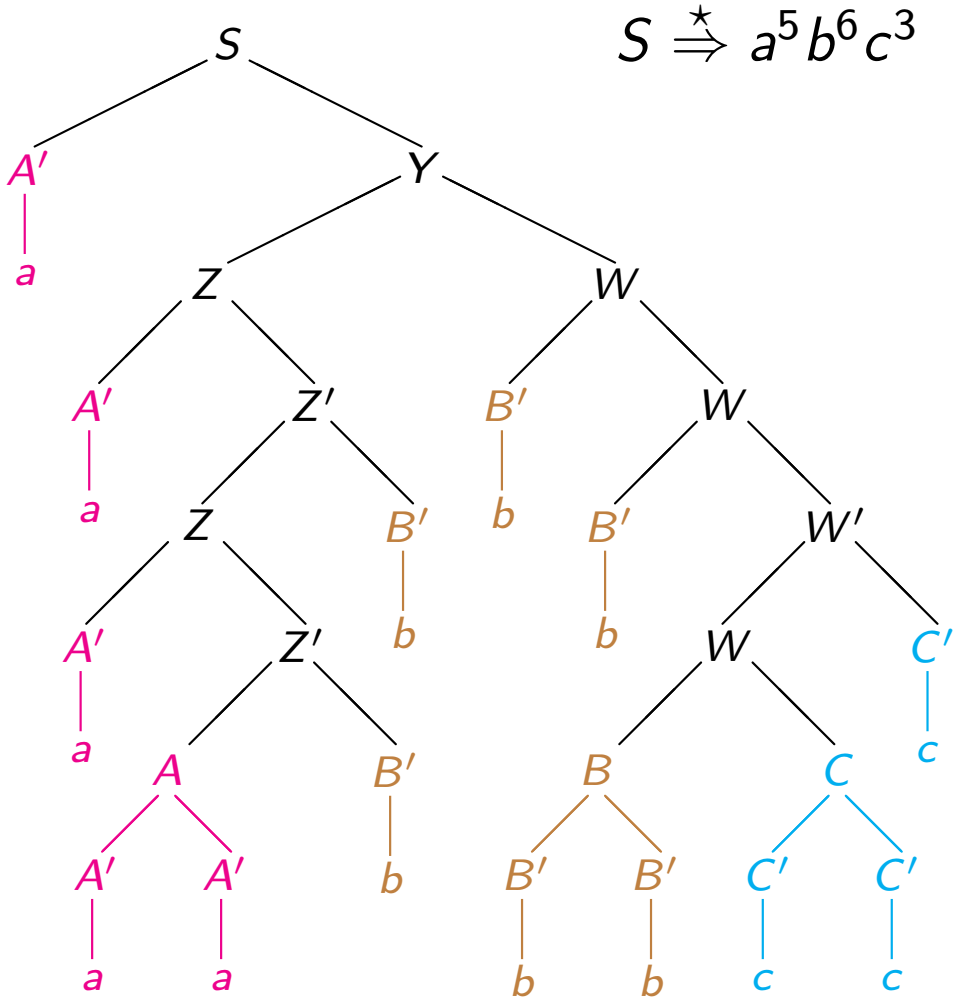

- Unary variables: A, A', B, B', C, C'
- $L_S, L_Y \subseteq a^+ b^* c^+$
- $L_Z, L_{Z'} \subseteq a^+ b^+$
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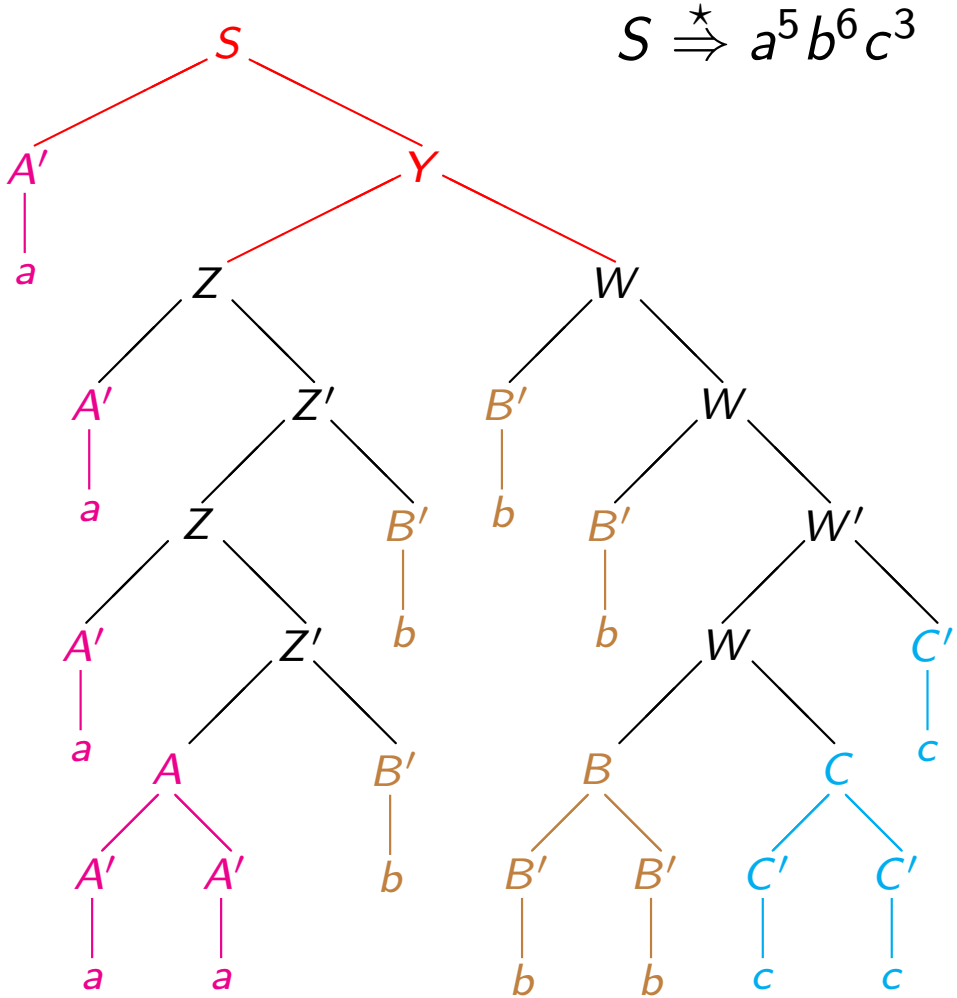

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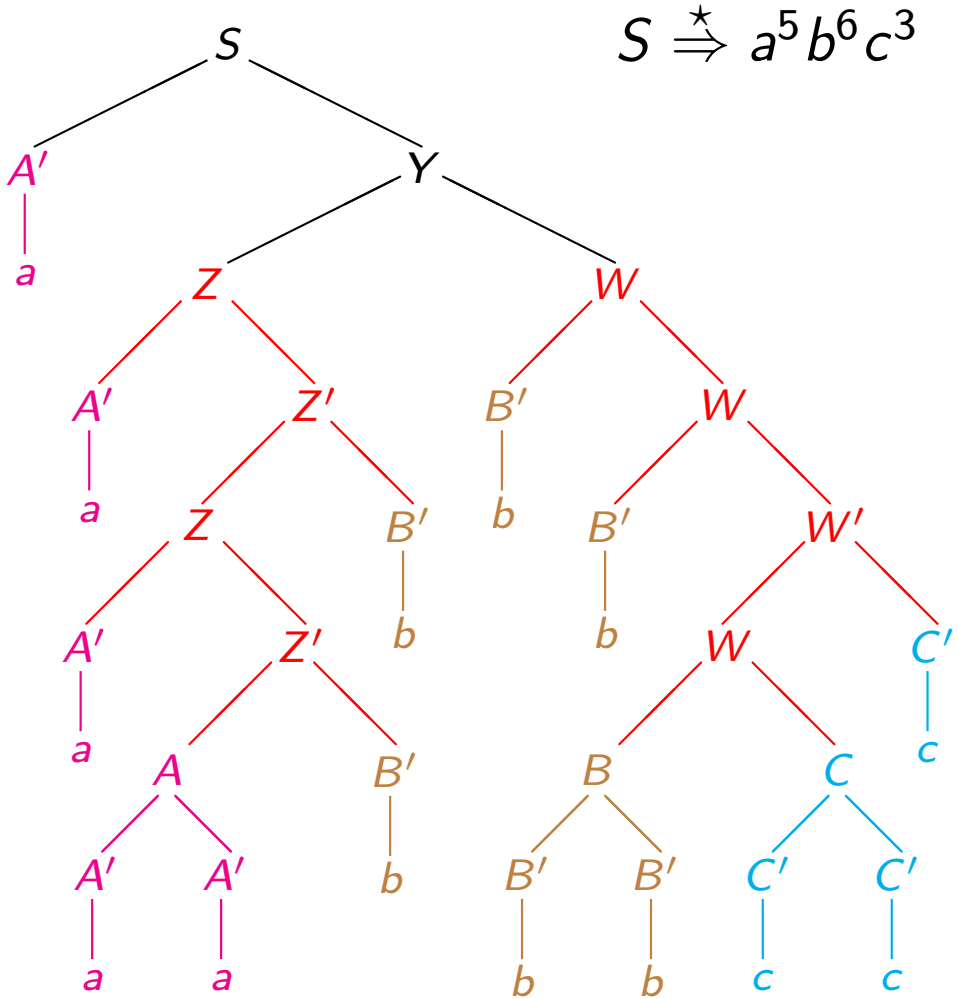

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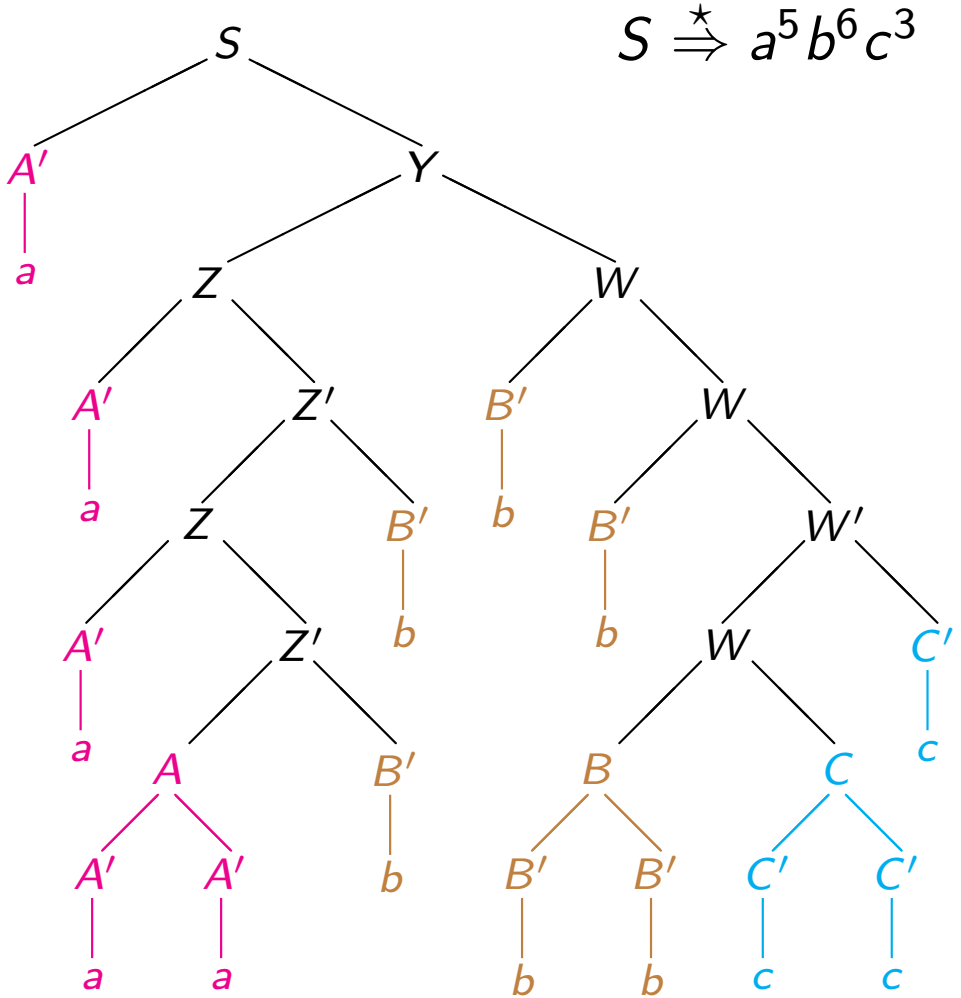
- $L_Z, L_{Z'} \subseteq a^+ b^+$
- $L_W, L_{W'} \subseteq b^+ c^+$

Example $\Sigma = \{a, b, c\}$



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by simulating a particular derivation from S

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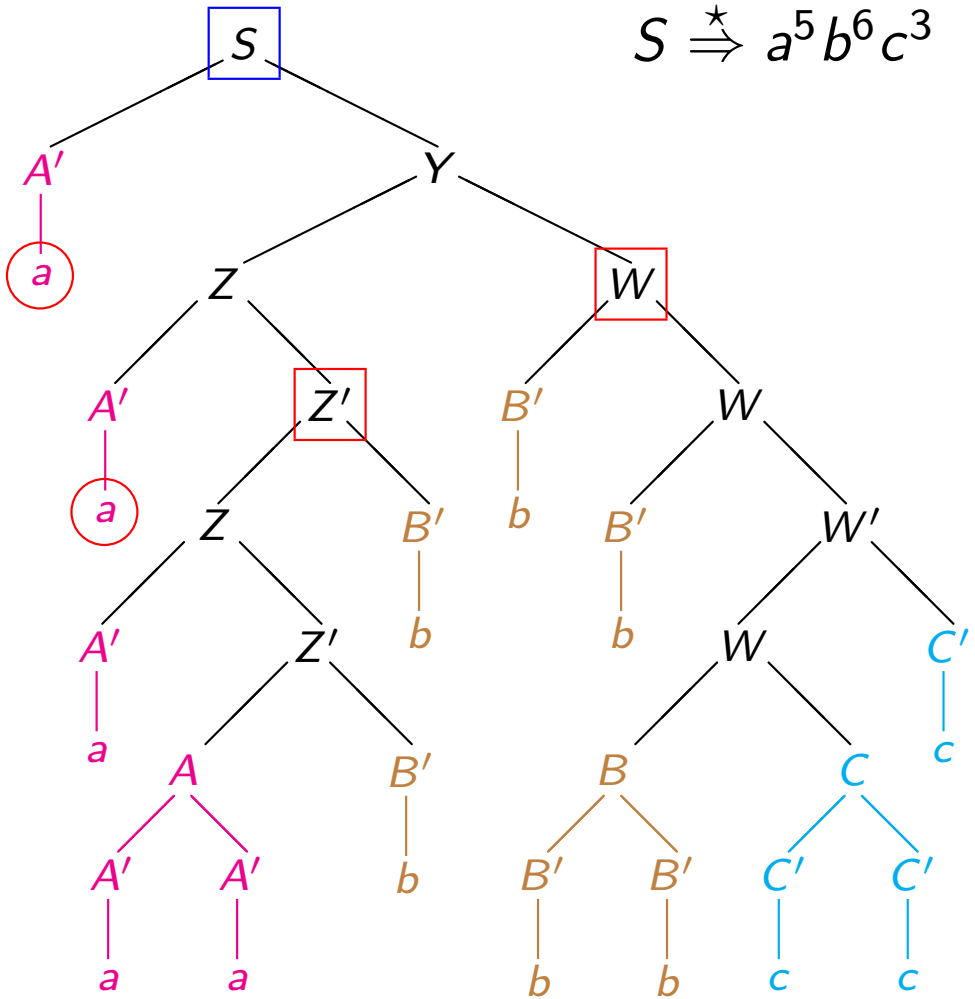

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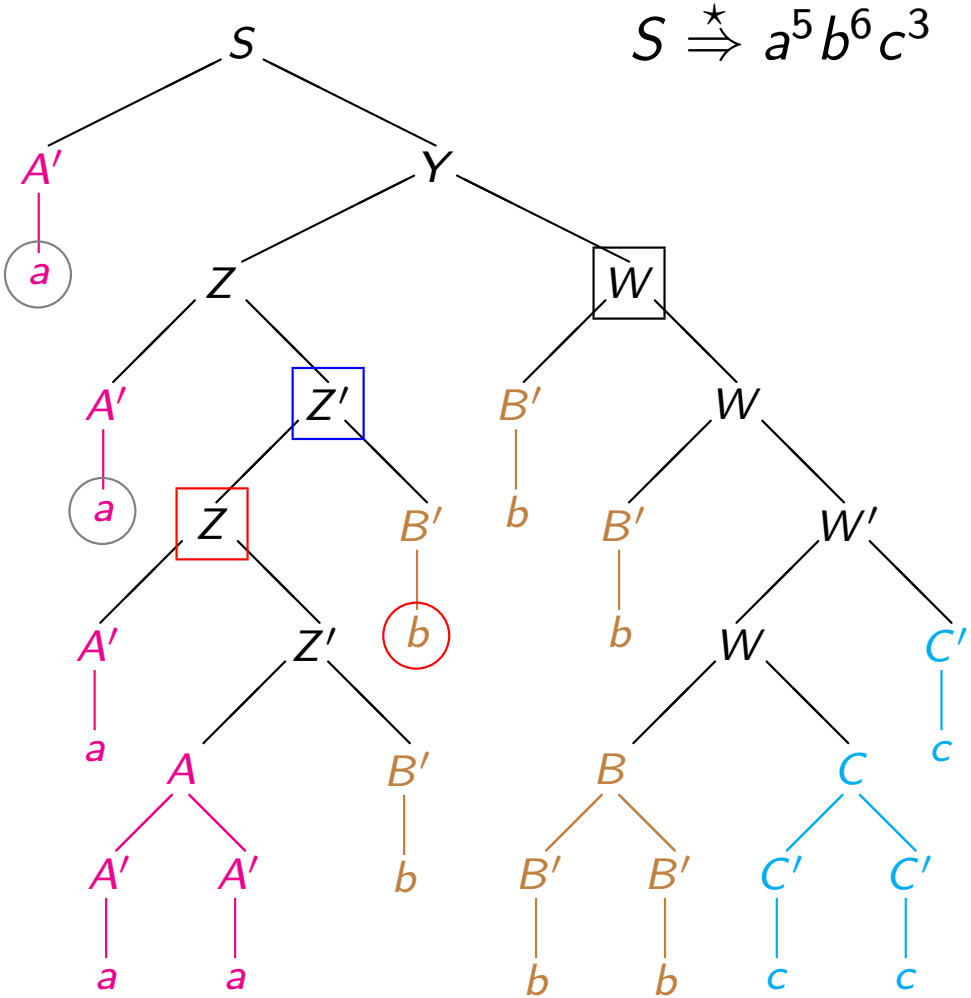

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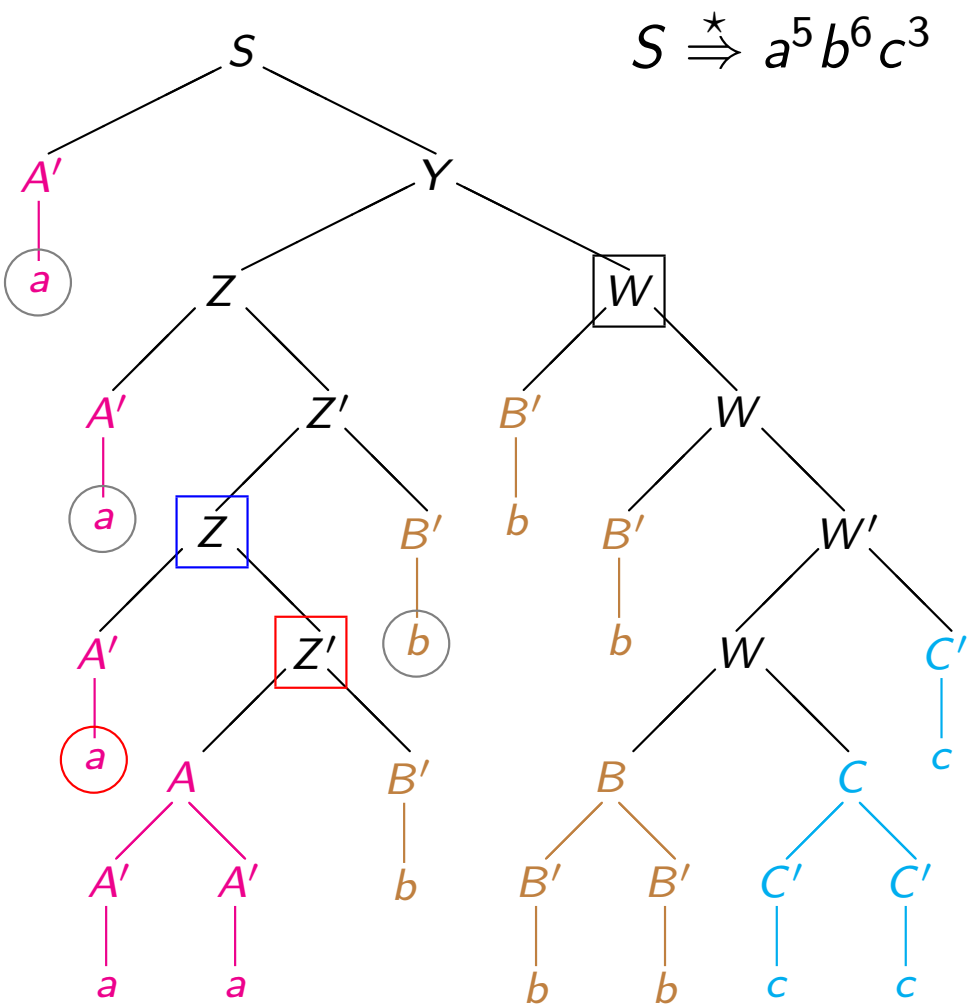

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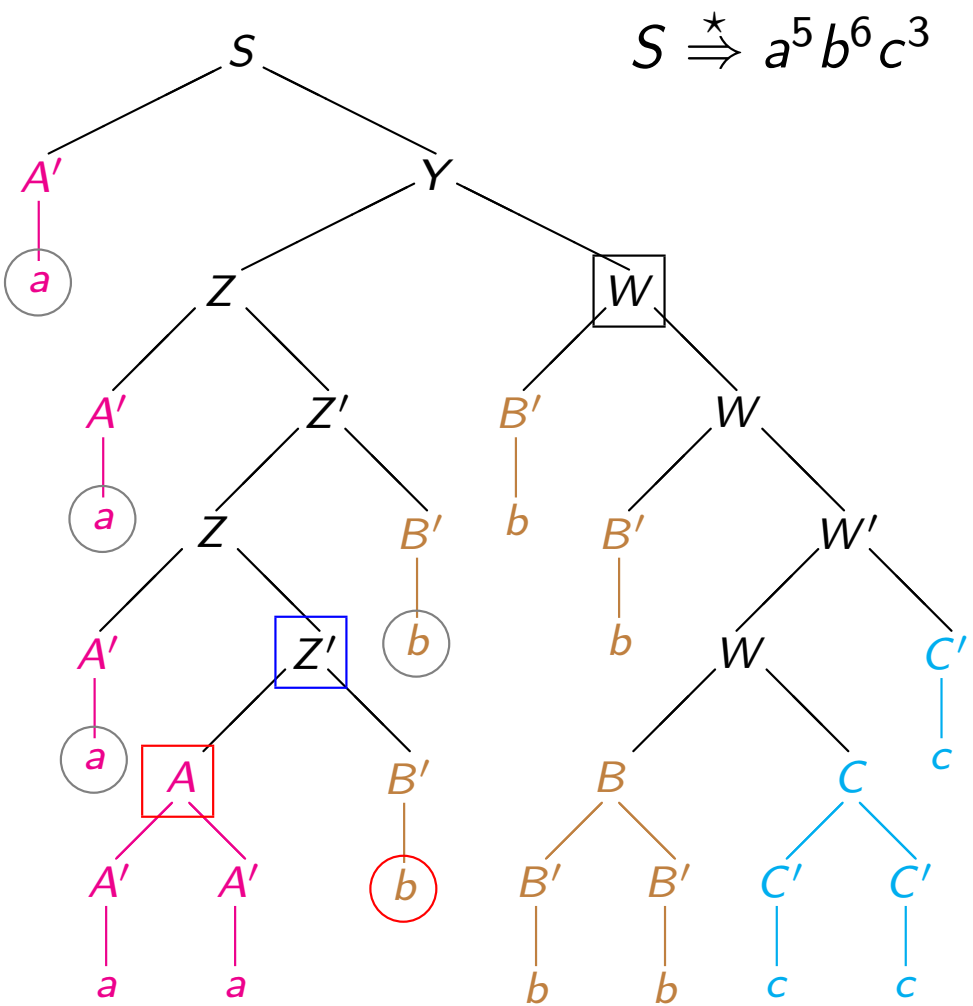

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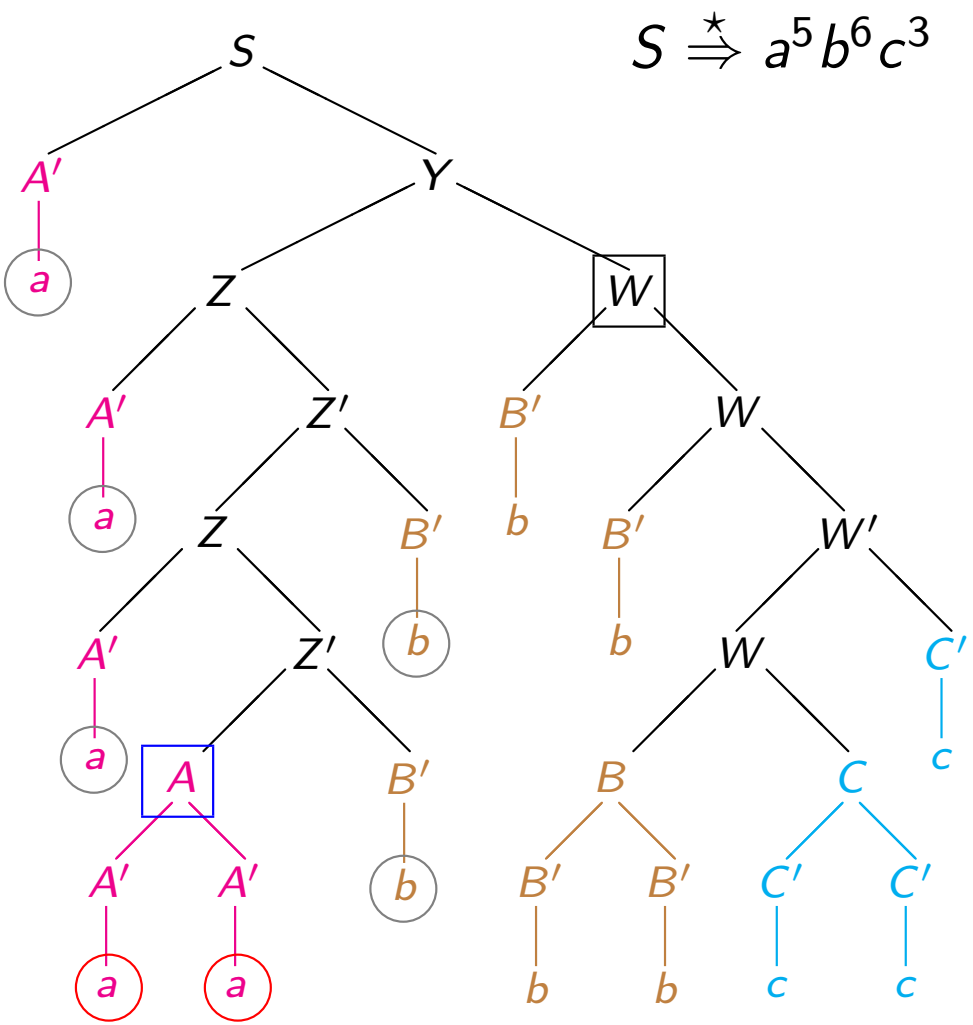

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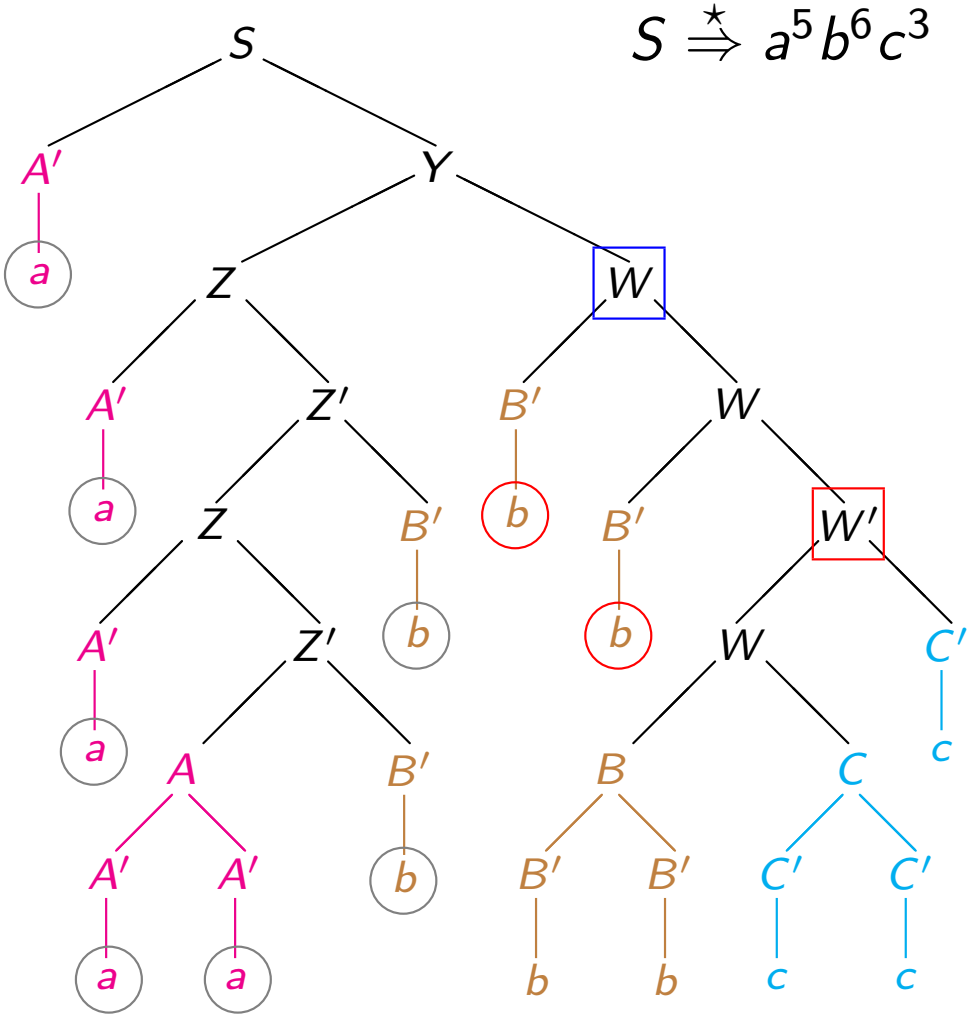

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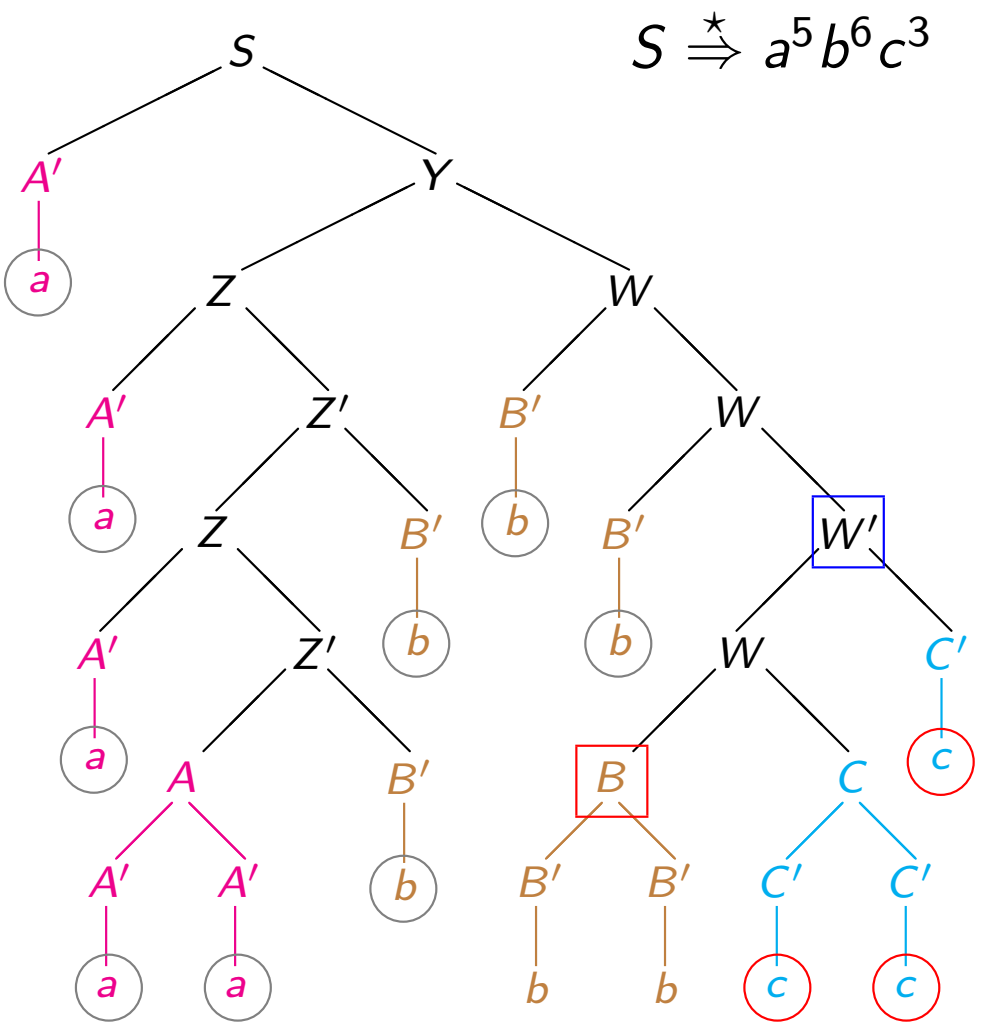

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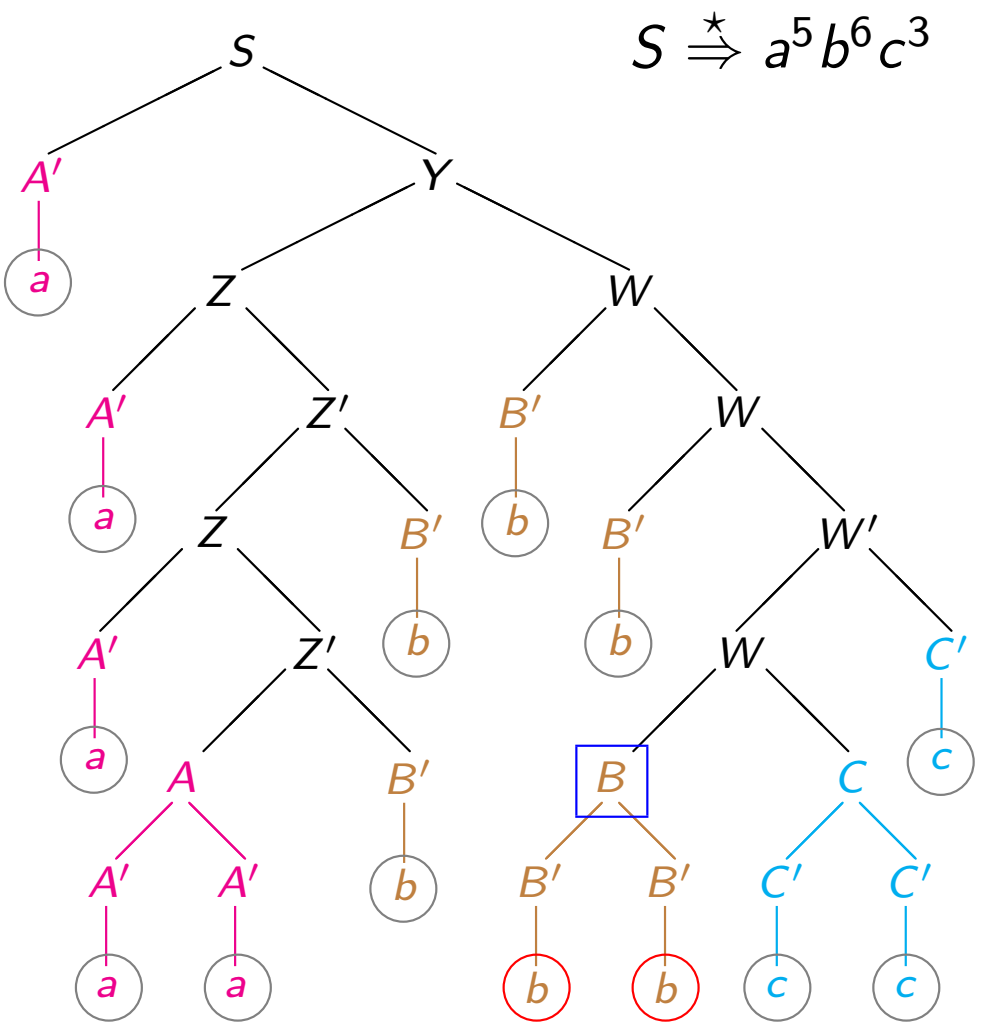

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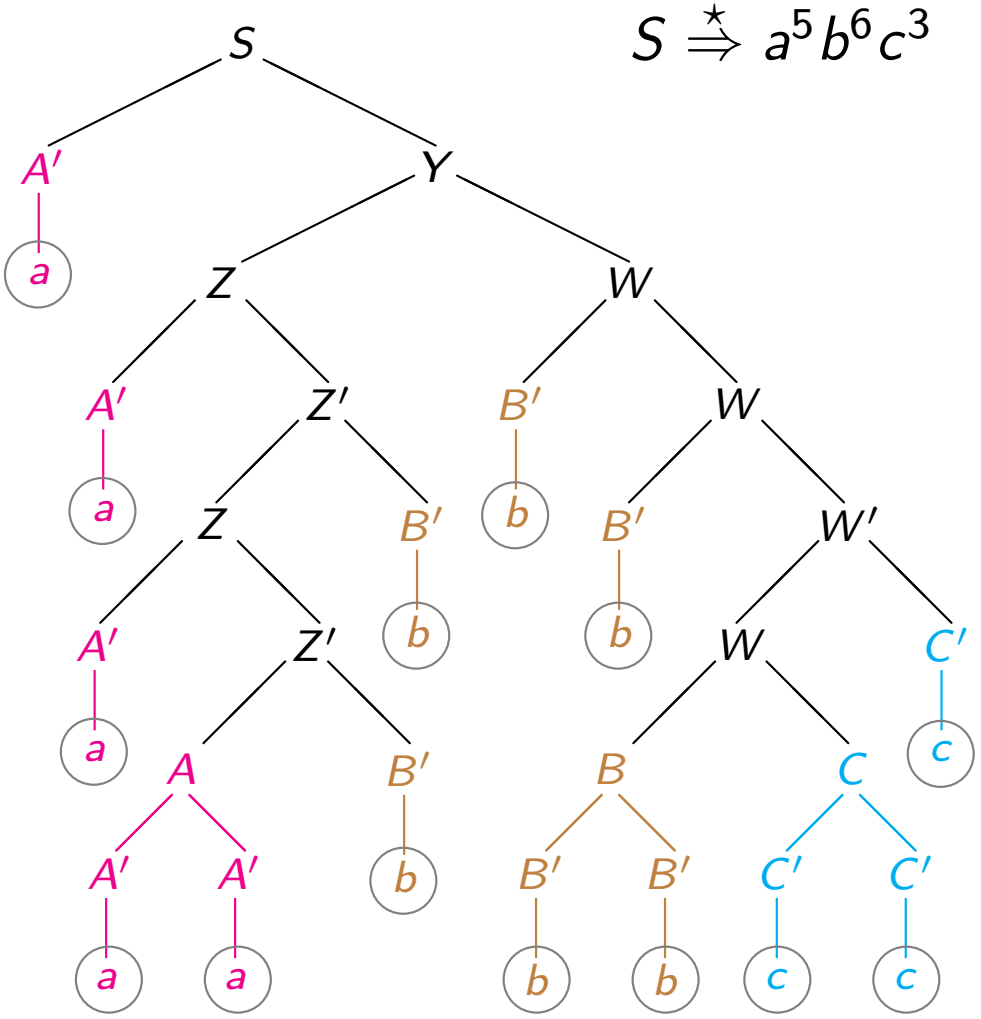

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- This derivation process is simulated by an automaton which tests the matching between generated terminals and input symbols
- At each step the automaton needs to remember at most $\#\Sigma - 1$ variables
- The process is nondeterministic
- It can be implemented using $O(h^{\#\Sigma-1})$ states
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Second Contribution: Binary Context-Free Languages

Theorem

Let G grammar in Chomsky normal form with h variables with a binary terminal alphabet.
Then there is a DFA A with at most $2^{h^{O(1)}}$ states s.t. $L(A) =_\pi L(G)$

The proof relies the following results:

Lemma ([Kopczyński&To '10])

For G as in the theorem, it holds that $\psi(L(G)) = \bigcup_{i \in I} Z_i$ where:

- *I is a set of indices with $\#I = O(h^2)$*
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From sets Z_i it is possible to derive "small" DFAs and, by standard constructions, the DFA A s.t. $L(A) =_\pi L(G)$

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all context-free languages?

- Bounded case
  crucial argument: it is enough to remember $\#\Sigma - 1$ variables
- Binary case
  the main lemma does not hold for alphabets with ≥ 3 letters

Other questions:

- What about *word bounded* CFLs?
  i.e., subsets of $w_1^* w_2^* \cdots w_p^*$ where each w_i is a string
- In our construction, the cost is double exponential in the size of the alphabet. State whether or not this is optimal.

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