Descriptional Complexity and Regular Languages

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Outline

Introduction: What is Descriptional Complexity?

The Question of Sakoda and Sipser

Unary Automata

Sadoka&Sipser Question vs L $\overset{?}{=}$ NL

Context-Free Grammars vs Regular Languages

Conclusion

A Classical Example: Deterministic vs Nondeterministic Automata

Formal language point of view:

- The class of languages recognized by NFAs coincides with the class of languages recognized by DFAs

Descriptional complexity point of view:

- Each n-state NFA can be simulated by a DFA *with* 2^n *states* [Rabin&Scott '59]
- For each integer n there exists a language L_n s.t.:
  - L_n is accepted by an n-state NFA
  - the minimum DFA for L_n requires 2^n states

  [Meyer&Fischer '71]
- Hence:

  The exact cost, in terms of states, of the simulation of NFAs by DFAs is 2^n

Descriptional Complexity

Given

- $\mathcal{C}$, a class of languages
- $\mathcal{S}$, a formal system (e.g., class of devices, class of grammars,...) able to represent all the languages in $\mathcal{C}$

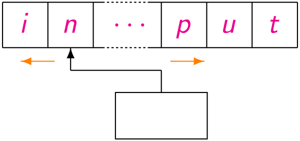
What is the *size* of the representations of the languages in $\mathcal{C}$ by the system $\mathcal{S}$?

Descriptional complexity compares different descriptions of a same class of languages:

- given $\mathcal{S}'$, another formal system able to represent all the languages in $\mathcal{C}$:

  Find the *relationships between the sizes* of the representations in the system $\mathcal{S}$ and in the system $\mathcal{S}'$ of the languages belonging to $\mathcal{C}$

Finite State Automata



Base version:

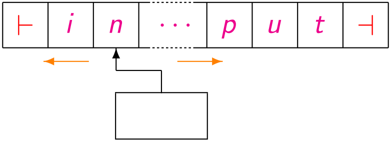
one-way deterministic finite automata (1DFA)

- one-way input tape
- deterministic transitions

Possibile variants allowing:

- nondeterministic transitions
  - one-way nondeterministic finite automata (1NFA)
- input head moving forth and back
  - two-way deterministic finite automata (2DFA)
  - two-way nondeterministic finite automata (2NFA)
- alternation
- ...

Two-Way Automata: Technical Details



- Input surrounded by the endmarkers ⊢ and ⊣
- Transition function $\delta : Q \times (\Sigma \cup \{\vdash, \dashv\}) \to 2^{Q\times\{-1,0,+1\}}$ where $-1, 0, +1$ are the possible movements of the input head
- $w \in \Sigma^*$ accepted iff there is a computation
  - with input tape $\vdash w \dashv$
  - from the initial state q_0, scanning the left endmarker ⊢
  - reaching a final state

1DFA, 1NFA, 2DFA, 2NFA

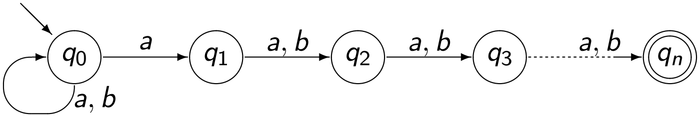
What about the power of these models?

They share the same computational power, namely they characterize the class of *regular languages*, however...

...some of them are more succinct

Example: $L = (a+b)^* a (a+b)^{n-1}$

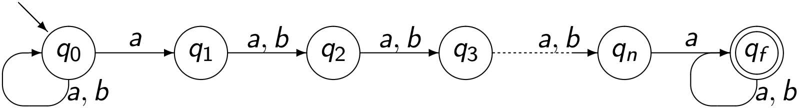
- L is accepted by a 1NFA with $n+1$ states



- The minimum 1DFA accepting L requires 2^n states
- We can get a *deterministic* automaton for L with $n+2$ states, which reverses the input head direction just one time
- Hence L is accepted by
  - a 1NFA and a 2DFA with approx. the same number of states
  - a minimum 1DFA exponentially larger

Example: $L = (a+b)^*a(a+b)^{n-1}a(a+b)^*$

- L is accepted by a 1NFA with $n+2$ states



- The minimum 1DFA accepting L uses $3 \cdot 2^{n-1} + 1$ states
- Using head reversals the number of states becomes linear
- Even in this case L is accepted by
  - a 1NFA and a 2DFA with linearly related numbers of states
  - a minimum 1DFA exponentially larger

Example: $L = (a+b)^*a(a+b)^{n-1}a(a+b)^*$

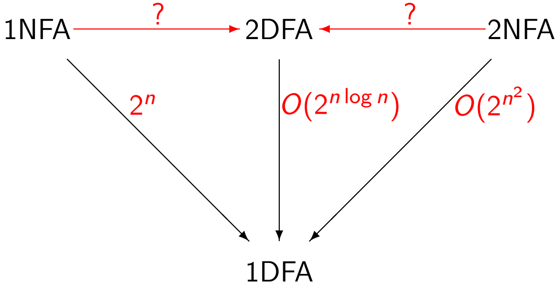
$\vdash$	b	b	a	b	a	a	b	a	a	a	$\dashv$

$n = 4$

```
while input symbol $\neq a$ do move to the right
move $n$ squares to the right
if input symbol $= a$ then accept
                  else move $n-1$ cells to the left
                       repeat from the first step
Exception: if input symbol $=\dashv$ then reject
```

- This can be implemented by a 2DFA with $O(n)$ states
- By a different algorithm, L can be also accepted by a 2DFA with $O(n)$ states *which changes the direction of its input head only at the endmarkers* (a *sweeping* automaton)

Costs of the Optimal Simulations Between Automata



[Rabin&Scott '59, Shepardson '59, Meyer&Fischer '71, . . .]

Question

How much the possibility of moving the input head forth and back is useful to eliminate the nondeterminism?

Costs of the Optimal Simulations Between Automata

$$\begin{tikzcd}
\text{1NFA} \arrow[r, "?"] \arrow[dr, "2^n"] & \text{2DFA} \arrow[d, "O(2^{n \log n})"] & \text{2NFA} \arrow[l, "?"'] \arrow[dl, "O(2^{n^2})"] \\
& \text{1DFA} &
\end{tikzcd}$$

Problem ([Sakoda&Sipser '78])

Do there exist polynomial simulations of

- *1NFAs by 2DFAs*
- *2NFAs by 2DFAs ?*

Conjecture

These simulations are not polynomial

Sakoda&Sipser Question: Lower Bounds

Polynomial lower bounds for the cost $c(n)$ of simulation of 1NFAs by 2DFAs:

- $c(n) \in \Omega(\frac{n^2}{\log n})$ [Berman&Lingas '77]
- $c(n) \in \Omega(n^2)$ [Chrobak '86]

Exponential lower bounds for the simulation of 2NFAs by 2DFAs, for special classes of resulting machines:

- sweeping automata [Sipser '80]
- oblivious automata [Hromkovič&Schnitger '03]
- "few reversal" automata [Kapoutsis '11]

Sweeping Automata

Definition (Sweeping Automata)

A two-way automaton A is said to be sweeping if and only if

- A is deterministic
- the input head of A can change direction only at the endmarkers

Each computation is a sequence of complete traversals of the input

- Sweeping automata can be exponentially larger than 1NFAs [Sipser '80]
- However, they can be also *exponentially larger* than 2DFAs [Berman '81, Micali '81]

“Few Reversal” Automata [Kapoutsis ’11]

Definition (Few Reversal Automata)

A two-way automaton A makes few reversals if and only if the number of reversals on input of length n is $o(n)$

Model between sweeping automata ($O(1)$ reversals) and 2NFAs

Theorem ([Kapoutsis ’11])

- *Few reversal DFAs can be exponentially larger than few reversal NFAs and, hence, than 2NFAs*
- *Sweeping automata can be exponentially larger than few reversal DFAs*
- *Few reversal DFAs can be exponentially larger than 2DFAs*

Hence, this result really extends Sipser’s separation, but does not solve the full problem

Sakoda&Sipser Question

Problem ([Sakoda&Sipser '78])

Do there exist polynomial simulations of

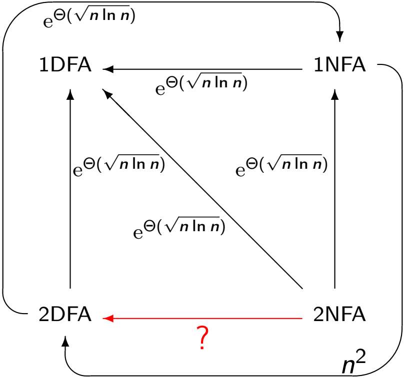
- *1NFAs by 2DFAs*
- *2NFAs by 2DFAs ?*

Another possible restriction:

The unary case $\#\Sigma = 1$

Optimal Simulation Between Unary Automata

The costs of the optimal simulations between automata are different in the unary and in the general case



[Chrobak '86,
Mereghetti&Pighizzini '01]

Optimal Simulation Between Unary Automata

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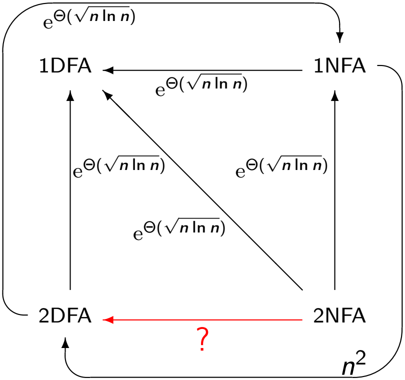
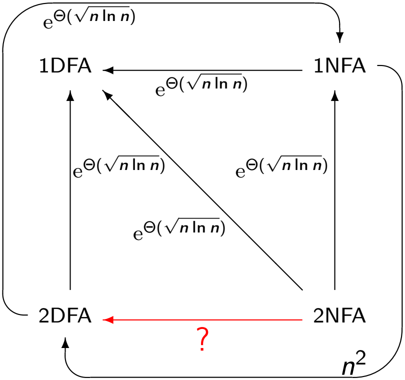


1NFA → 2DFA
In the unary case
this question is solved!
(polynomial conversion)

Optimal Simulation Between Unary Automata

The costs of the optimal simulations between automata are different in the unary and in the general case



2NFA → 2DFA
Even in the unary case
this question is open!

- $e^{\Theta(\sqrt{n \ln n})}$ upper bound
  (from 2NFA → 1DFA)
- $\Omega(n^2)$ lower bound
  (from 1NFA → 2DFA)

A better upper bound $e^{O(\ln^2 n)}$
has been proved!

Sakoda&Sipser Question: Current Knowledge

- Upper bounds

	1NFA→ 2DFA	2NFA→ 2DFA
unary case	$O(n^2)$ optimal	$e^{O(\ln^2 n)}$
general case	exponential	exponential

Unary case [Chrobak '86, Geffert Mereghetti&Pighizzini '03]

- Lower Bounds
  In all the cases, the best known lower bound is $\Omega(n^2)$ [Chrobak '86]

Unary Case: Quasi Sweeping Automata

[Geffert Mereghetti&Pighizzini '03]

In the study of unary 2NFA, sweeping automata with some *restricted nondeterministic capabilities* turn out to be very useful:

Definition

A 2NFA is quasi sweeping (qsNFA) iff both

- nondeterministic choices and head reversals

are possible only at the endmarkers

Theorem (Quasi Sweeping Simulation)

Each n-state unary 2NFA A can be transformed into a 2NFA M s.t.

- *M is quasi sweeping*
- *M has at most $N \leq 2n + 2$ states*
- *M and A are "almost equivalent"*
  (differences are possible only for inputs of length $\leq 5n^2$)

Quasi Sweeping Simulation: Consequences

Several results using quasi sweeping simulation of unary 2NFAs have been found:

(i) Subexponential simulation of unary 2NFAs by 2DFAs
Each unary n-state 2NFA can be simulated by a 2DFA with $e^{O(\ln^2 n)}$ states [Geffert Mereghetti&Pighizzini '03]

(ii) Polynomial complementation of unary 2NFAs
Inductive counting argument for qsNFAs [Geffert Mereghetti&Pighizzini '07]

(iii) Polynomial simulation of unary 2NFAs by 2DFAs *under the condition* L = NL

(iv) Polynomial simulation of unary 2NFAs by unambiguous 2NFAs (unconditional)

We are going to discuss (iii) and (iv) [Geffert&Pighizzini '10]

Logspace Classes and Graph Accessibility Problem

L: class of languages accepted in logarithmic space by *deterministic* machines

NL: class of languages accepted in logarithmic space by *nondeterministic* machines

Problem

$\mathsf{L} \stackrel{?}{=} \mathsf{NL}$

Graph Accessibility Problem GAP

- Given $G = (V, E)$ oriented graph, $s, t \in V$
- Decide whether or not G contains a path from s to t

Theorem ([Jones'75])

GAP *is complete for* NL

Hence GAP $\in$ L iff L $=$ NL

Polynomial Deterministic Simulation (under L = NL)

Outline

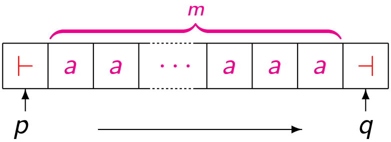
From now on, we fix an n-state unary 2NFA A

- We give a *reduction* from $L(A)$ to GAP
  i.e, for each input string a^m we define a graph $G(m)$ s.t.

$$a^m \in L(A) \iff G(m) \in \mathrm{GAP}$$

- *Under the hypothesis* L = NL
  this reduction will be used to build 2DFA equivalent to A,
  with a number of states polynomial in n
- Actually we do not work directly with A:
  we use the qsNFA M obtained from A
  according to the quasi sweeping simulation

The graph $G(m)$



Given the qsNFA M with N states and an input a^m the graph $G(m)$ is defined as:

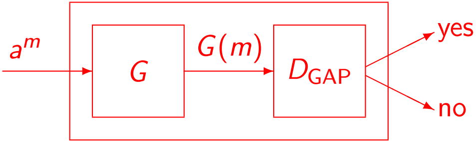
- the vertices are the states of M
- (p, q) is an edge iff M can traverse the input
  - from one endmarker in the state p
  - to the opposite endmarker in the state q
  - without visiting the endmarkers in the meantime

Then

$a^m \in \mathrm{L(M)}$ iff $G(m)$ contains a path from q_0 to q_F

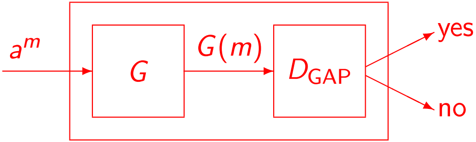
The existence of the edge (p, q) can be verified by a finite automaton $A_{p,q}$ with N states

Polynomial Deterministic Simulation (under L = NL)



- Suppose L = NL
- Let D_{GAP} be a logspace bounded *deterministic* machine solving GAP
- On input a^m, compute $G(m)$ and give the resulting graph as input to D_{GAP}
- This decides whether or not $a^m \in L(M)$

Polynomial Deterministic Simulation (under L = NL)



- The graph $G(m)$ has N vertices, the number of states of M
- D_{GAP} uses space $O(\log N)$
- M is fixed. Hence N is constant, independent on the input a^m

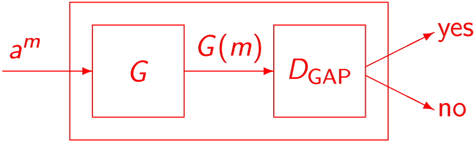
  The worktape of D_{GAP} can be encoded in a finite control using a number of states polynomial in N

- The graph $G(m)$ can be represented with N^2 bits

  Representing the graph in a finite control would require exponentially many states

- To avoid this, input bits for D_{GAP} are computed "on demand"

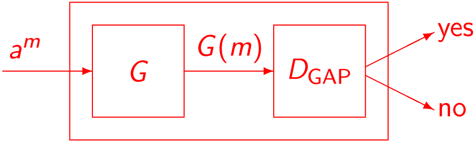
Polynomial Deterministic Simulation (under L = NL)



We define a unary 2DFA M' equivalent to M

- M' keeps in its finite control:
  - The input head position of D_{GAP}
  - The worktape content of D_{GAP}
  - The finite control of D_{GAP}
- This uses a number of states polynomial in N

Polynomial Deterministic Simulation (under L = NL)



We define a unary 2DFA M' equivalent to M

- On input a^m, M' simulates D_{GAP} on input $G(m)$
- Input bits for D_{GAP} are the entries of $G(m)$ adjacency matrix
- Subroutine $A_{p,q}$, using N states, computes the input bit corresponding to (p, q)
- Considering all possible (p, q), this part uses at most N^3 states

Summing Up...

We described the following simulation:

- M is *almost equivalent* to the original 2NFA A
- Hence, M' is *almost equivalent* to A
- Possible differences for input length $\leq 5n^2$
- They can be fixed in a preliminary scan ($5n^2 + 2$ more states)
- The resulting automaton has polynomially many states

A	given unary 2NFA	n states
$\Downarrow$		Quasi Sweeping Simulation
M	qsNFA almost equivalent to A	$N \leq 2n + 2$ states
$\Downarrow$		Deterministic Simulation
M'	2DFA equivalent to M	$poly(N)$ states
$\Downarrow$	Preliminary scan to accept/reject inputs of length $\leq 5n^2$ then simulation of M' for longer inputs	
M''	2DFA equivalent to A	$poly(n)$ states

Polynomial Deterministic Conditional Simulation

Theorem ([Geffert&Pighizzini '10])

If L = NL *then each n-state unary 2NFA can be simulated by an equivalent 2DFA with a polynomial number of states*

Hence

> *Proving the Sakoda&Sipser conjecture for unary 2NFAs would separate* L *and* NL

Another condition:

Theorem ([Berman&Lingas '77])

If L = NL *then there exists a polynomial p s.t.*
for each $m > 0$ and k-state 2NFA A,
there exists a $p(mk)$-state 2DFA A' s.t.
$L(A') \subseteq L(A)$ *and* $L(A) \cap \Sigma^{\leq m} = L(A') \cap \Sigma^{\leq m}$

Further relationships with logspace complexity in [Kapoutsis '11]

What About the Converse?

Question

Does a polynomial simulation of unary 2NFAs by 2DFAs imply $\mathsf{L} = \mathsf{NL}$*?*

- The answer is positive, *under an additional assumption*:

  The transformation from unary 2NFAs to 2DFAs
  must be computable in deterministic logspace

- Under this assumption, the answer is positive *even restricting* to the simulation of unary 1NFAs by 2DFAs:

Theorem

If there exists a deterministic logspace bounded transducer transforming each n-state unary 1NFA into an equivalent $n^{O(1)}$*-state 2DFA then* $\mathsf{L} = \mathsf{NL}$

Unambiguous Logspace (Nonuniform)

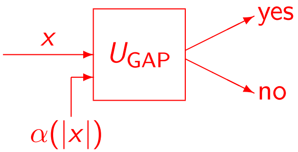
Theorem ([Reinhardt&Allender '00])

$\mathsf{NL} \subseteq \mathsf{UL}/\mathit{poly}$

- UL/*poly*
  class of languages accepted by *unambiguous* logspace machines with a *polynomial advice*, i.e.,
- A sequence of strings $\{\alpha(n) \mid n \geq 0\}$ of polynomial length
- With each input string x, the machine also receives the advice string $\alpha(|x|)$

Corollary

$\mathsf{GAP} \in \mathsf{UL}/\mathit{poly}$



Making Unary 2NFAs Unambiguous

Theorem ([Geffert&Pighizzini '10])

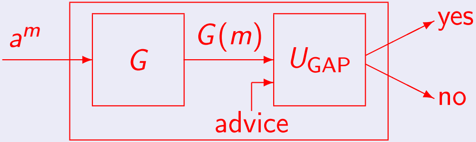
Each n-state unary 2NFA can be simulated by an equivalent unambiguous 2NFA with a polynomial number of states

Proof.

- Similar to the polynomial deterministic conditional simulation
- Hypothetical machine D_{GAP} replaced with U_{GAP} and advice

Given a 2NFA the size of $G(m)$ (input of U_{GAP}) is fixed

- Hence the advice is fixed (i.e., it does not depend on a^m)
- Advice encoded in the hardware of the simulating machine



Descriptional Complexity of Regular Languages

- Different variants of finite automata characterize regular languages
- However, we can describe regular languages using more powerful formalisms or devices, as context-free grammars and pushdown automata

What about the sizes of CFGs or PDAs describing regular languages vs the sizes of finite automata?

Descriptional Complexity Measures

- Context-free grammars:
  number of variables?

  For $n \geq 1$, consider the language $L_n = (a^n)^*$:
  - L_n requires n states to be accepted by 1DFAs or 1NFAs
  - L_n is generated by the grammar with one variable S and the productions

    $$S \to a^n S \qquad S \to \epsilon$$

- Thus, the number of variables cannot be a descriptional complexity measure for context-free grammars.
- However, for grammars in *Chomsky Normal Form* the number of variables is a "reasonable" measure of complexity [Gruska '73]

Context-Free vs Regular: Descriptional Complexity

Given a context-free grammar of size n, generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer&Fischer '71])

For any recursive function f and arbitrarily large integers n, there exists a CFG of size n generating a regular language L, s.t. any DFA accepting L must have at least $f(n)$ states

Then:

the trade-off between CFG and finite automata is not recursive

However...

the witness language is defined over a binary alphabet

What about unary languages?

CFGs vs Automata: Unary Case

Theorem ([Ginsburg&Rice, '62])

Every unary context-free language is regular

What about descriptional complexity?

Theorem ([Pighizzini Shallit&Wang '02])

Given a unary CFG in Chomsky normal form with h variables, there exist:

- *an equivalent 1NFA with at most $2^{2h-1} + 1$ states*
- *an equivalent 1DFA with at most 2^{h^2} states*

These bounds are tight, namely, matching lower bound have been obtained.

Final considerations

- Many results in formal language theory have been revisited and refined considering descriptional complexity aspects
- Having descriptions of small size can be very interesting for many applications
  (e.g., small circuits and programs for portable devices)
- There are strong connections between descriptional complexity and structural complexity
  (e.g., Sakoda and Sipser question and L vs NL question, machine simulations, similar techniques as crossing sequences, inductive counting, Savitch simulation,...)
- Other complexity measure deserve further investigation
  (e.g., ambiguity degrees, measures of nondeterminism)

Final considerations

- We discussed only a few aspects related to descriptional complexity of regular languages
- Many other aspects have been investigated
- Probably the first paper in the area:
  A.R. Meyer, M.J. Fischer:
  Economy of Description by Automata, Grammars, and Formal Systems, FOCS 1971, 188–191
- An intersting survey:
  J. Goldstine, M. Kappes, C.M.R. Kintala, H. Leung,
  A. Malcher, D. Wotschke:
  Descriptional Complexity of Machines with Limited Resources,
  J. Universal Comp. Science 8(2): 193–234 (2002)
- Annnual international worskhop on descriptional complexity

  Descriptional Complexity of Formal Systems (DCFS)

  Limburg, Germany, July 25-27, 2011.