Pairs of Complementary Unary Languages with "Balanced" Nondeterministic Automata

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A general problem

Compare the number of the states of *complementary automata*, i.e, automata accepting a regular language and its complement:

Given an n-state automaton accepting L, how many states are necessary and sufficient to accept L^c?

- Deterministic automata (DFAs): trivial.
- Nondeterministic automata (NFAs):
  - trivial upper bound: 2^n, optimal [Birget 1993]
  - differences between the general case and the case of *unary languages*. [Mera&Pighizzini 2005]

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Size of complementary NFAs: general vs. unary case

There are languages having "small" complementary NFAs:

For each integer n there exists a regular language L such that:

- L is accepted by an n-state NFA,
- L^c is accepted by an NFA with at most $n+1$ states,
- the minimum DFA accepting L requires 2^n states.

Hence:

- L is a witness of the maximal state gap between NFAs and DFAs,
- the gap between the total size of smallest NFAs accepting L and L^c and corresponding DFAs is exponential.

The language L is defined over a two letter alphabet.

The unary case looks completely different:

If a unary language L has a "small" NFA then each NFA accepting L^c must be "large".

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Unary case $\Sigma = \{a\}$

The cost of the optimal simulation of n-state NFAs by DFAs in the unary case reduces from 2^n to the function $F(n) = e^{\Theta(\sqrt{n \cdot \ln n})}$, which is subexponential but superpolynomial. [Chrobak 1986]

However:

If L is a unary language accepted by an n-state NFA s.t. the minimum equivalent DFA requires $F(n)$ states, then also *each* NFA *accepting* L^c *requires at least* $F(n)$ *states.* [Mera&Pighizzini 2005]

In other words:

- If L is a witness of the maximal state gap between unary NFAs and equivalent DFAs then each NFA for L^c must have as many states as the minimum DFA.
- Hence, taking into account the total number of states of smallest NFAs accepting L and L^c, the superpolynomial gap with the size of DFAs disappears.

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Paper contributions

How large can be the gap between the total size of pairs of NFAs accepting a unary language and its complement and the minimum DFA ?

We prove that this gap is *superpolynomial*, not too far from $F(n)$:

There are infinitely many unary languages L such that:

- the state gap between NFAs and DFAs accepting L is a little bit smaller than $F(n)$, but it is still superpolynomial,
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We also prove superpolynomial gaps between:

- the sizes of unary *unambiguous automata* and of DFAs,
- the sizes of unary *self-verifying automata* and of DFAs.

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$L \subseteq \{a\}^*$ is said to be *cyclic* iff it is accepted by a DFA whose transition graph is just one loop.

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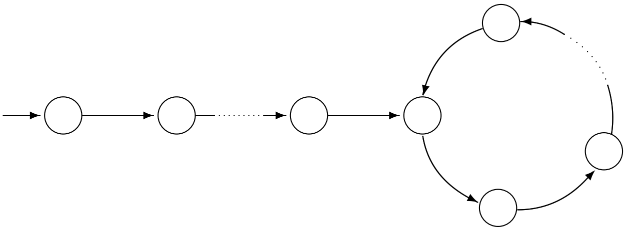

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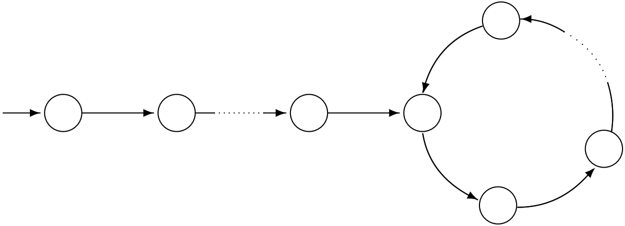

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Let us consider:

- $\lambda_0, \lambda_1, \ldots, \lambda_{s-1}$ a sequence of $s \geq 1$ powers of different primes,
- the smallest integer $\hat{s} \geq s$ dividing one λ_ℓ, $\ell \in \{0, \ldots, s-1\}$.

We define the language:

$$L = \bigcup_{i=0}^{s-1} \{a^k \mid k \bmod \hat{s} = i = k \bmod \lambda_i\}.$$

How to recognize L and L^c?

The witness languages

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To decide if an input string a^k belongs to L we can use the following procedure:

- $i \leftarrow k \bmod \hat{s}$
- if $i = 0$ then accept iff $k \bmod \lambda_0 = 0$
- if $i = 1$ then accept iff $k \bmod \lambda_1 = 1$
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- if $i = s - 1$ then accept iff $k \bmod \lambda_{s-1} = s - 1$
- if $i \geq s$ then reject

Nondeterministic version:

- guess i, with $0 \leq i < s$
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The nondeterministic version can be implemented by an NFA A^+.

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NFA A^+:

- one initial state and s disjoint loops
- in the initial state one among s possible transitions is chosen: guess i, with $0 \le i < s$
- transition i leads to the ith loop, which implements: accept iff $k \bmod \hat{s} = i$ and $k \bmod \lambda_i = i$

length of the ith loop: $\mathrm{lcm}(\hat{s}, \lambda_i)$

Size of A^+:

- Summing up: total number of states: $1 + \sum_{i=0}^{s-1} \mathrm{lcm}(\hat{s}, \lambda_i)$
- However $\mathrm{lcm}(\hat{s}, \lambda_i) = \begin{cases} \hat{s} \cdot \lambda_i, & \text{if } i \neq \ell; \\ \lambda_\ell, & \text{otherwise.} \end{cases}$
- Hence, the total number of the states is:

$$N = 1 + \lambda_\ell + \hat{s} \cdot \sum_{i=0, i\neq\ell}^{s-1} \lambda_i .$$

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- However $\mathrm{lcm}(\hat{s}, \lambda_i) = \begin{cases} \hat{s} \cdot \lambda_i, & \text{if } i \neq \ell; \\ \lambda_\ell, & \text{otherwise.} \end{cases}$
- Hence, the total number of the states is:

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An NFA for $L = \bigcup_{i=0}^{s-1}\{a^k \mid k \bmod \hat{s} = i = k \bmod \lambda_i\}$

NFA A^+:

- one initial state and s disjoint loops
- in the initial state one among s possible transitions is chosen:
  guess i, with $0 \leq i < s$
- transition i leads to the ith loop, which implements:
  accept iff $k \bmod \hat{s} = i$ and $k \bmod \lambda_i = i$

  length of the ith loop: $\mathrm{lcm}(\hat{s}, \lambda_i)$

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An example

Let $\lambda_0 = 2^2, \lambda_1 = 5$. Then $\hat{s} = s = 2$.

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\begin{aligned}
L &= \{a^k \mid (k \bmod 2 = 0 = k \bmod 2^2) \vee (k \bmod 2 = 1 = k \bmod 5)\} \\
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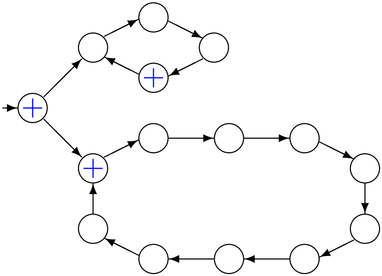

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How to accept the complement of L

Deterministic procedure to decide whether or not $a^k \in L$:

- $i \leftarrow k \bmod \hat{s}$
- if $i = 0$ then accept iff $k \bmod \lambda_0 = 0$
- if $i = 1$ then accept iff $k \bmod \lambda_1 = 1$
- ...
- if $i = s - 1$ then accept iff $k \bmod \lambda_{s-1} = s - 1$
- if $i \geq s$ then reject

To recognize L^{c} we just need to replace “accept” with ”reject” and vice versa, in the previous procedure.

How to accept the complement of L

Deterministic procedure to decide whether or not $a^k \in L^{\mathrm{c}}$:

- $i \leftarrow k \bmod \hat{s}$
- if $i = 0$ then reject iff $k \bmod \lambda_0 = 0$
- if $i = 1$ then reject iff $k \bmod \lambda_1 = 1$
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Nondeterministic version:

- guess i, with $0 \leq i < s$
- if $i < s$ then accept iff $k \bmod \hat{s} = i$ and $k \bmod \lambda_i \neq i$
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Hence:

$$L^{\mathrm{c}} = \bigcup_{i=0}^{s-1} \{a^k \mid k \bmod \hat{s} = i \wedge k \bmod \lambda_i \neq i\} \cup \{a^k \mid k \bmod \hat{s} \geq s)\}$$

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Same transition graph as A^+:

- For $i = 0, \ldots, s-1$, the ith loop is used to accept if $k \bmod \hat{s} = i$ and $k \bmod \lambda_i \neq i$
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A^+ and A^- have the same size $N = 1 + \lambda_\ell + \hat{s} \cdot \sum_{i=0, i\neq\ell}^{s-1} \lambda_i$.

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The previous example: how to accept the complement

Let $\lambda_0 = 2^2, \lambda_1 = 5$. Then $\hat{s} = s = 2$.

$$L = \{a^k \mid \text{if } k \text{ is even then } k \bmod 2^2 = 0, \text{ if } k \text{ is odd then } k \bmod 5 = 1\}.$$

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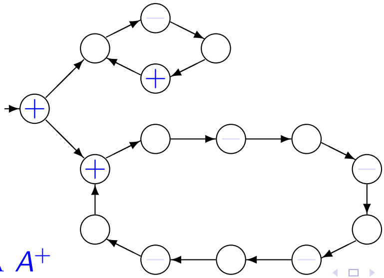

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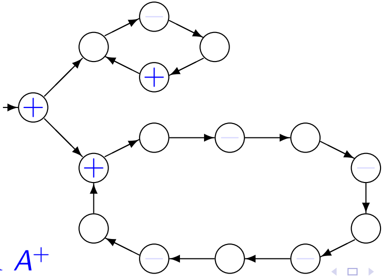

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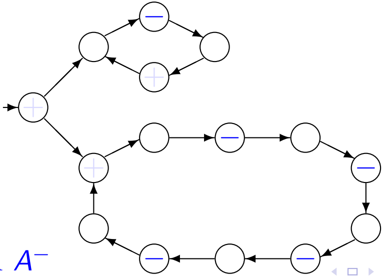

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By previous discussion and by investigating the structure of the minimum DFA accepting L, we proved that:

Theorem

- *The transition graph of the minimum* DFA *A accepting L is a loop of $\lambda_0 \cdot \lambda_1 \cdots \lambda_{s-1}$ states.*
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We now study the following question:

How large is the gap between N and $\lambda_0 \cdot \lambda_1 \cdots \lambda_{s-1}$,
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The witness languages and their automata

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- Initially investigated in group theory. [Landau 1903, 1909]
- Fundamental role in the analysis of simulations among various models of unary automata. [Chrobak 1986]

Definition

For a positive integer n:

$$F(n) = \max\{\mathrm{lcm}(\lambda_0, \ldots, \lambda_{s-1}) \mid \lambda_0 + \cdots + \lambda_{s-1} = n\},$$

where $\lambda_0, \ldots, \lambda_{s-1}$ denote arbitrary positive integers.

- Sharp estimation: [Szalay 1980]

$$F(n) = e^{(1+o(1)) \cdot \sqrt{n \cdot \ln n}}$$

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- If $\lambda_i = a \cdot b$ with $\gcd(a, b) = 1$ then $a + b \leq a \cdot b$:
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Hence:

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Witness language and Landau Function

Let us fix an integer n and some powers $\lambda_0, \ldots, \lambda_{s-1}$ of different primes such that:

$$\lambda_0 + \cdots + \lambda_{s-1} \leq n \quad \text{and} \quad F(n) = \lambda_0 \cdots \lambda_{s-1}.$$

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- some properties of $F(n)$ [Nicolas 1968, Grantham 1995],
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The exponential gap

Summing up, for infinitely many N we provided a language L s.t.:

- L and L^{c} are accepted by two NFAs A^{+} and A^{-} using at most N states,
- the minimum DFA accepting L (or L^{c}) must use at least $e^{\Omega(\sqrt[3]{N \cdot \ln^2 N})}$ states.

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The gap between the total number of states of the pair of complementary unary NFAs A^{+}, A^{-} and the minimum equivalent DFA is superpolynomial.

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The state set is partitioned in three groups:

- *accepting* states ("yes")
- *rejecting* states ("no")
- *neutral* states ("I do not know")

It is required that:

- on each input string at least one accepting or one rejecting state is reached,
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SVFAs characterize the class of regular languages.

Each n-state SVFA can be converted into an equivalent DFA with $O(3^{n/3}) \approx O(1.45^n)$ states. This cost is tight, *for an input alphabet of at least two letters*. [Jirásková & Pighizzini 2009]

What about the tight cost in the unary case?

- It must be strictly smaller than $F(n)$, the cost of the conversion of unary NFAs into DFAs. [J&P 2009]
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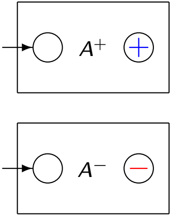
From pairs of complementary NFAs to a SVFAs

Let A^+ and A^- be two complementary NFAs.
We can build an equivalent SVFA A as the "union" of A^+ and A^-, using a new initial state:



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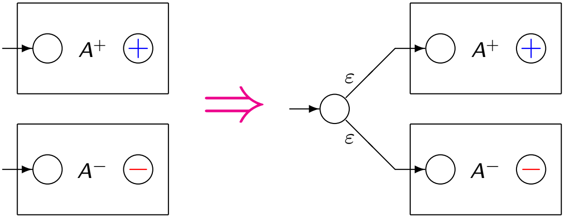

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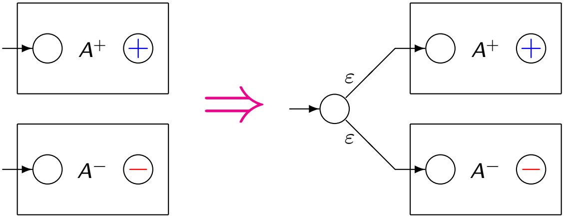

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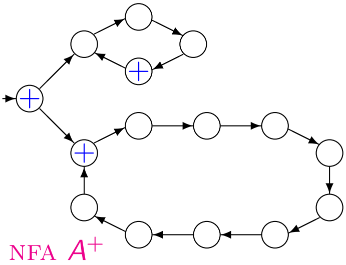
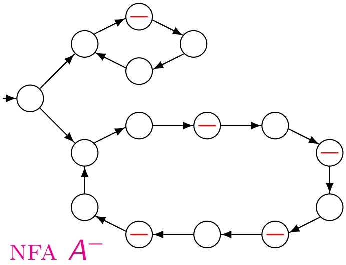
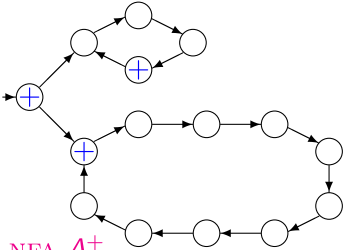

NFA A^+


NFA A^-

From our witness NFAs to SVFAs



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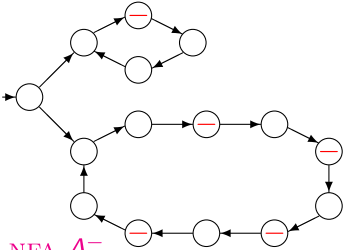

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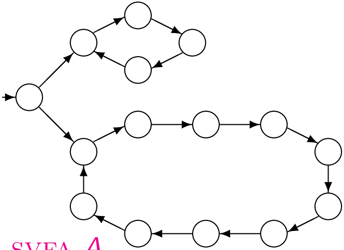

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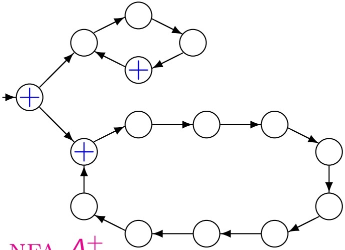

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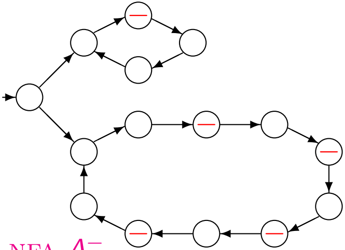
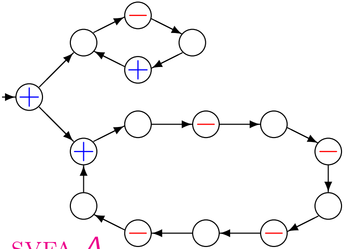


NFA A^-



SVFA A

Same size!

Conclusion

We proved that the following gaps are superpolynomial, *even for unary cyclic languages:*

- between the total size of two NFAs accepting one language and its complement and the size of the corresponding DFA,
- between the sizes of SVFAs and DFAs.

The witness NFAs A^+ and A^- we used are *unambiguous.*
Hence, also the following gaps are superpolynomial for unary cyclic languages:

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The superpolynomial function in all these gaps is $e^{\Omega(\sqrt[3]{N \cdot \ln^2 N})}$.
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