

Deterministic Pushdown Automata and Unary Languages

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Outline of the talk

- Context-free grammars and pda's vs regular languages: some descriptonal complexity results
- Exponential simulation of unary dpda's by dfa's
- Optimality of the simulation
- Conclusions

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Context-free vs regular: descriptive complexity

Given a context-free grammar (or a pushdown automaton) of size n , generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer and Fischer, 1971])

For any recursive function f and arbitrarily large integers n , there exists a cfg G of size n generating a regular language L , s.t. any dfa accepting L must have at least $f(n)$ states.

As a consequence, the trade-off between context-grammars and finite automata is not recursive.

However... The witness language L is defined over a binary alphabet.

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Unary languages

$$\Sigma = \{a\}$$

Theorem ([Ginsurg and Rice, 1962])

Every unary context-free language is regular.

Hence the classes of unary regular languages and unary *context-free* languages coincide!

Problem

Study the equivalence between unary context-free and regular languages from the descriptive complexity point of view.

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Unary case [Pighizzini, Shallit, Wang, 2002]

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For any cfg in Chomsky normal form with h variables, generating a unary language, there exists an equivalent dfa with 2^{h^2} states. Furthermore, this bound is tight.

Corollary

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- Each dpda of size s accepting a regular language can be simulated by a dfa with $2^{2^{2^s}}$ states. [Stearns, 1967]
- This upper bound was reduced to 2^{2^s} in [Valiant, 1975].
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Number of its states

- Size of a pushdown automaton:

Total number of symbols needed to write down its description.

We have to keep into account:

- the number of the states
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- the length of the strings that can be pushed in one move on the stack
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- We can prove that each dpda of size s can be converted into an equivalent dpda in normal form such that the product of
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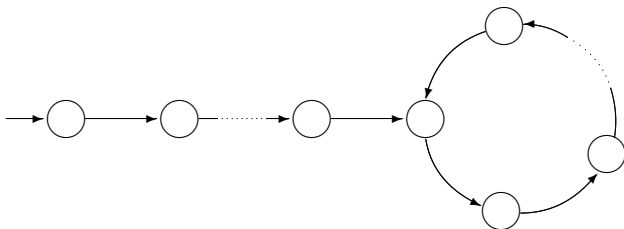
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Unary dfa's

Input alphabet $\Sigma = \{a\}$



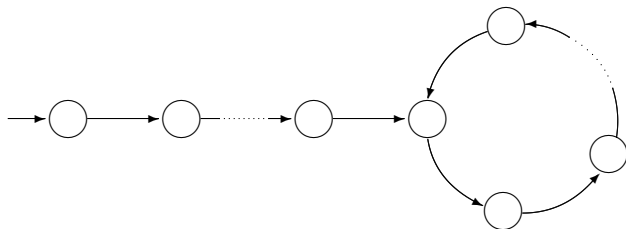
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$L \subseteq \{a\}^*$ is regular iff $\exists \mu \geq 0, \lambda \geq 1$ s.t.

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L.$$

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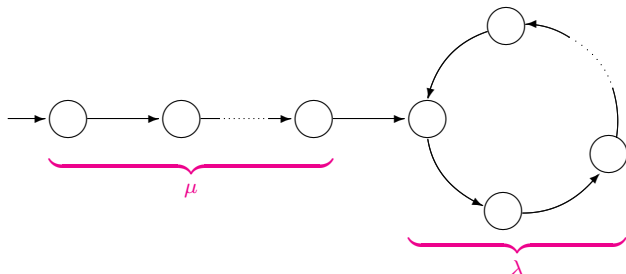
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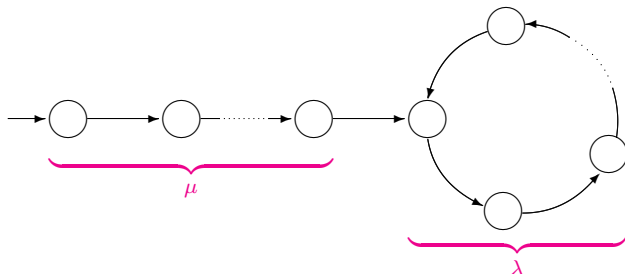
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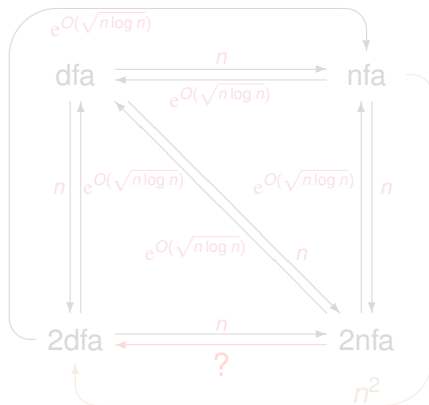
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Unary automata

The costs of the optimal simulations between automata are different in the unary and in the general case!

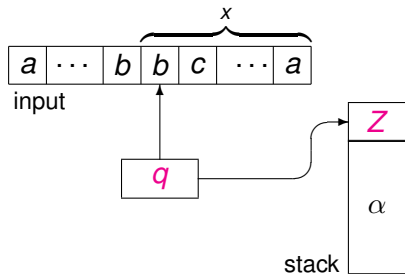
Costs in the unary case:

[Chrobak 1986, Mereghetti and Pighizzini 2001]



Pushdown automata

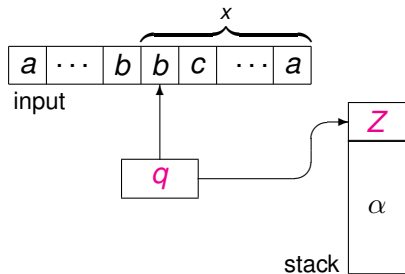
$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$



- $(q, x, Z\alpha)$ configuration
- $[qZ]$ mode

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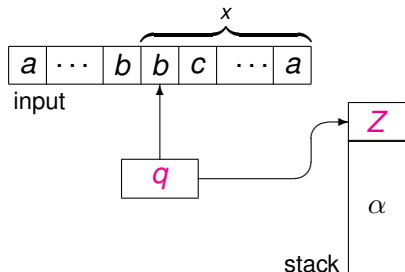
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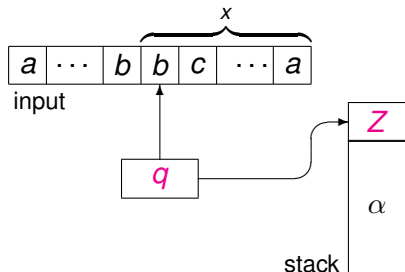
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 - if $\delta(q, \epsilon, Z) \neq \emptyset$ then $\delta(q, a, Z) = \emptyset$, for each $a \in \Sigma$
 - $\#\delta(q, \sigma, Z) \leq 1$, for each $\sigma \in \Sigma \cup \{\epsilon\}$.
- **Deterministic cfl's:** acceptance by *final states*

$$L(M) = \{x \in \Sigma^* \mid (q_0, x, Z_0) \vdash^* (q, \epsilon, \gamma), q \in F, \gamma \in \Gamma^*\}$$

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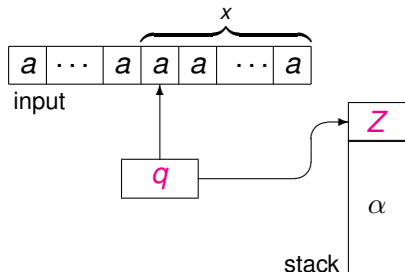
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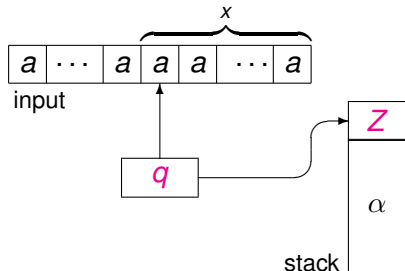
Unary deterministic pda's:

For each integer $t \geq 0$:

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$[qA] \leq [pB]$ iff all the following conditions hold:

- 1. A configuration C with mode $[qA]$ is reachable from the initial configuration
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- 3. If a configuration C' with mode $[pB]$ is reachable before C , then $C' \leq C$

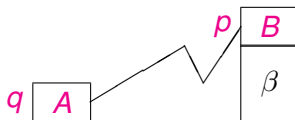
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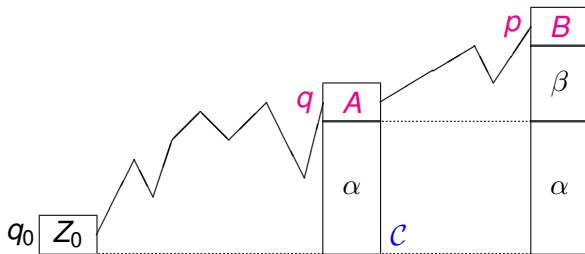
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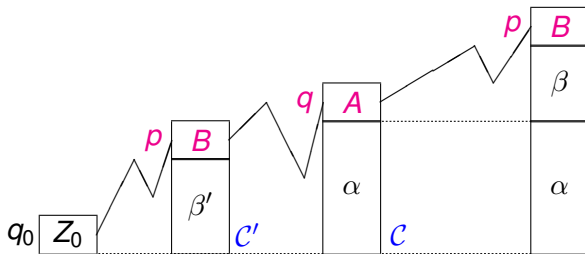
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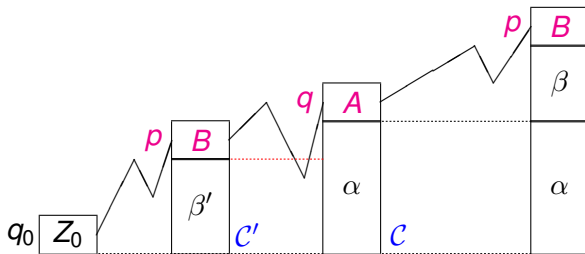
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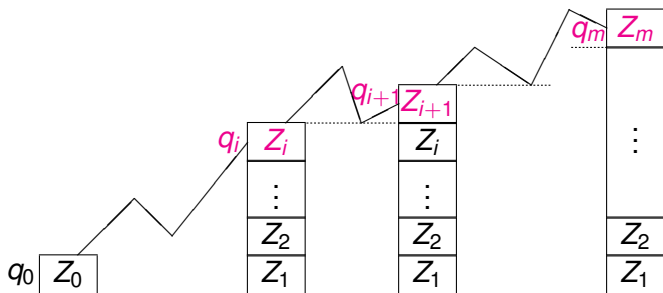
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Lemma

The relation \leq defines a partial order on the set of the modes.

h_t history at the time t

Stack content + state information



History h_t at the time t :

sequence of modes $[q_m Z_m] \dots [q_1 Z_1]$ s.t.:

- $Z_m \dots Z_1$ is the stack content after t computation steps
- for $i = 1, \dots, m$, $[q_i Z_i]$ is the mode of the last visited configuration with stack height i

Histories and modes

For each step $t \geq 0$ we consider:

- h_t history
- m_t mode (leftmost element of h_t)

For the given dpda M we consider:

- $H = \{h_t \mid t \geq 0\}$, the set all *reachable histories*
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Two possibilities:

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- H is finite
- The given dpda can be simulated by a deterministic automaton A whose set of states is H
- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \dots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \dots \leq [q_m Z_m]$
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- The histories in H grow in a periodic way, i.e.:
 $\exists \mu \geq 0, \lambda \geq 1, \exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:

$$h_t = \tilde{h}_{t \bmod \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$

- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ , from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
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Unary dpda's vs dfa's

As a consequence:

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Each unary dpda of size s can be simulated by a dfa with $2^{O(s)}$ states.

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Unary dpda's vs dfa's: lower bound

Given $s > 0$ consider $L_s = (a^{2^s})^*$.

We can prove that:

- There exists a dpda of size $8s + 4$ accepting L_s .
- Each dfa accepting L_s must have at least 2^s states.

Hence our simulation is optimal!

Problem: Does it is possible to reduce the cost of the simulation of unary dpda's, by using *nondeterministic* or *two-way* finite automata?

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Languages with “complex” dpda’s

Unary dpda’s can be exponentially more succinct than dfa’s.
Does this is true for *each* unary regular language?

Problem

For $m \geq 0$, let $L_m \subseteq a^*$ be a language accepted by a dfa with 2^m states.

Does there exists an equivalent dpda with $O(m)$ states?

The answer to this question is negative:

For each $m > 0$ there exists a language $L_m \subseteq a^*$ s.t.:

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w_m **de Bruijn word** of order m on $\{0, 1\}$:

- $|w_m| = 2^m + m - 1$
- each string of length m is a factor of w_m , occurring in w_m exactly one time
- the suffix and the prefix of length $m - 1$ of w_m coincide.

Example: $w_3 = 0001011100$

$L_m = \{a^k \mid \text{the letter of } w_m \text{ in position } k \text{ MOD } 2^m \text{ is } 1\}$,
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- M : a dpda of size s accepting L_m
- M' : M extended with an output tape to generate the de Bruijn word
- A : a dfa with $m + 1$ states, input alphabet $\{0, 1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'' : a dpda of size $O(ms)$, composition of M' and A , accepting $\{a^{2^m+m-1}\}$
- G : cfg grammar of size $O(ms)$, obtained from M'' , generating $\{w_m\}$

Lemma (Grossi, 2002)

The size of each grammar G generating $\{w_m\}$ must be at least $c \frac{2^m}{m}$, for some constant c .

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As a consequence we get the following *lower bound*:

Corollary

There exists a constant $K > 0$ such that the conversion of unary n -state dfa's into equivalent dpda's produces dpda's of size at least $K \frac{n}{\log^2 n}$, for infinitely many n 's.

Pda's vs cfg's

To prove the last result we have investigated the transformation of unary dpda's into context-free grammars.

- Each pda can be transformed into an equivalent cfg with $(\#Q)^2 \cdot \#\Gamma + 1$ variables.
- This number cannot be reduced, even if the given pda is deterministic [Goldstine, Price, Wotschke, 1962].

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Related questions and results

Bounded languages:

Subsets of $w_1^* w_2^* \dots w_n^*$, for given words w_1, \dots, w_n .

Extend the investigation to *bounded deterministic context-free* languages:

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In the nondeterministic case we have the following:

Theorem (Pigeonhole Principle)

Each bounded context-free language generated by a cfg with h variables in Chomsky normal form is accepted by a finite-turn pda with 2^h and $O(1)$ stack symbols.

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